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“S-shaped” Economic Dynamics. The Logistic and Gompertz curves generalized

Gloria Jarne   Julio Sánchez-Chóliz   Francisco Fatás-Villafranca

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Keywords: “S-shaped” curves, Logistic equation, Gompertz equation, Economic growth.

JEL: C61, E37, O30, O39

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1. INTRODUCTION

1.1 A brief historical review

S-shaped or sigmoid curves have often been used in the fields of demographics, biology and economics, in the first two cases to describe the evolution of populations and in the last to model processes of dissemination and self-organization associated with the spread of new technologies and products, technological change and, in general, economic growth.

The curves are used because of their ability to describe these processes and display their typical phases: emergent, inflexion and saturation. The first involves strong accelerated growth; the second, approximately linear growth; and the third, strong decelerated growth (see Foster and Wild (1999) p. 754). Without doubt, the expressive power of the curves' graphic representation has favored their application.

The use of these curves in demographics arose as a reaction to Malthus' exponential growth model, which could not properly explain the empirical data. The first sigmoids to be used, and for a long time the only ones, were the logistic and Gompertz curves, which are respectively expressed as follows:

$$\dot{x} = a\,x\,(c - x), \text{ where } a > 0 \text{ and } c > 0, \quad (1)$$

$$\dot{x} = a\,x\,(\ln c - \ln x), \text{ where } a > 0 \text{ and } c > 0. \quad (2)$$

Gompertz' original work was presented at the Royal Society in London in 1825 and is described in Smith and Keyfitz (1977). The logistic curve was applied for the first time by Verhulst, who published his research in 1838 in the journal "Correspondence Mathematique et Physique". Almost a century later, in 1920, R. Pearl and L.J. Reed rediscovered the logistic curve in the course of their study of the evolution of fly populations.

The first use of a sigmoid curve to analyze the economic growth is attributed to the French sociologist Gabriel Tarde, more specifically, in relation to innovation (see Tarde (1903)). Tarde's ideas were followed up by other scholars. Prescott (1922) obtained demand forecasts for automobiles using a Gompertz curve, whilst Kuznets (1930) and Burns (1934) studied long-term growth using the logistic and Gompertz curves. Ryan and Gross (1943) apply S-shaped diffusion to the case of new types of seeds and Levitt (1965) does so to the study of the "product life cycle". In Fisher and Pry (1971) we find a logistic diffusion innovation model based on the analogy between epidemic spread and information circulation. Sigmoid models very similar to the above are those of Blackman (1973), Floyd (in Bright (1968)) and Sharif and Kabir (1976).

Curves of this kind have also been used in more recent research. Thus, Englmann (1994) holds that a set of successive techno-economic paradigms can be identified in economic history and assumes that the learning curve in each paradigm can be described through a logistic model. Reati (1998) uses both the logistic and Gompertz' law to model the spread of technological revolutions. Foster and Wild (1999) show how the logistic equation can be used for modeling growth curves in the presence of a self-organizational change. Aoki and Yoshikawa (2002) capture the role of demand as a mechanism limiting growth itself, using logistic curves to describe the time utility of the new products. Finally, Metcalfe (2003) obtains a logistic dynamic in his diffusion model.

The extensive use made of sigmoid curves is undeniable. Some scholars have gone so far as to suggest that almost all social phenomena obey a sigmoid growth law, except during certain brief periods; see Montroll (1978).

1.2 Limitations of the logistic and Gompertz curves in empirical research

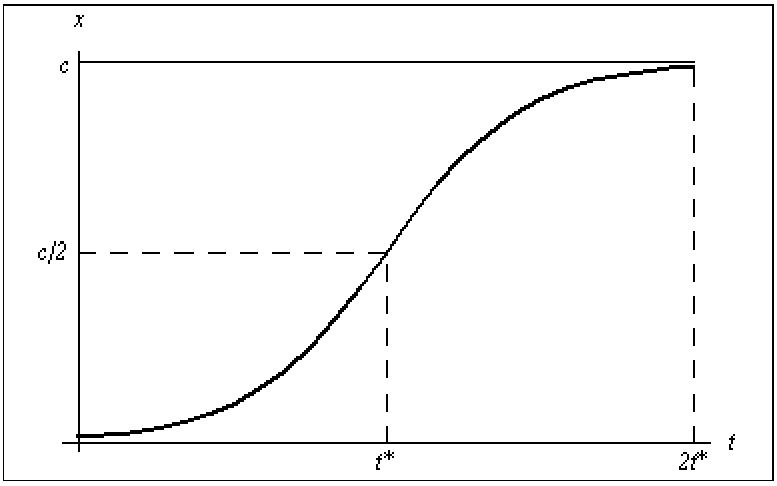
As will be apparent from the above references, the application of these curves in theoretical models has been highly effective (see, for example, the use Goodwin (1990) makes of them to obtain chaotic cycles and the demand analysis of Aoki and Yoshikawa (2002)). Nevertheless, empirical results have generally been less satisfactory for a number of reasons which we will now consider.

As we will make clear later, both the logistic and the Gompertz curves are characterized by constantly declining growth rates. Thus, although the curves are S-shaped, their growth rates never increase, as can be seen in Figures 1 and 2. Nevertheless, the empirical data tell us that periods of expansion (associated with an increasing growth rate) and contraction (decreasing growth rate) follow each other in economic growth processes. In this regard, see the NBER series or recall the popular recession criterion, that is to say, a declining GDP growth rate in two consecutive quarters. Moreover, both Schumpeter (1939) and his followers Fels (1964), Mensch (1979), Van Duijin (1985), Zarnowitz (1992) and Freeman (1996) identify a growth wave involving the acceleration or deceleration of growth and showing a certain bell-like evolution in the growth rates.

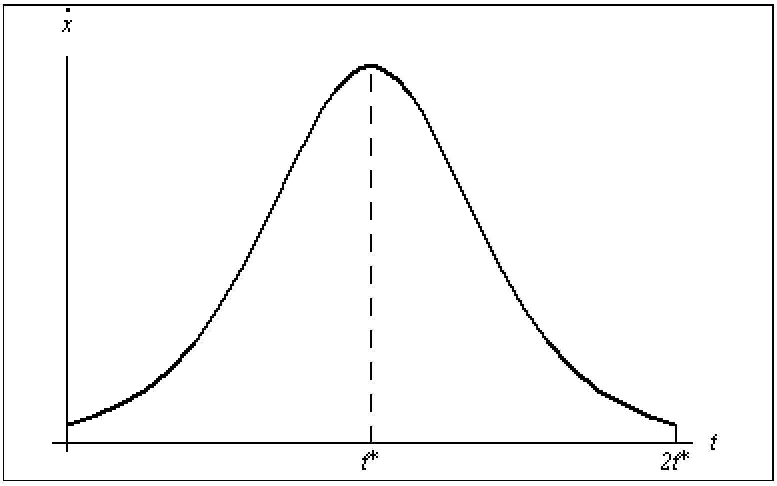
Figure 1: Representations of the conventional logistic dynamic

($a = 2$, $c = \sqrt{e}$, $x_0 = 0.02$)

Solution path



Growth rhythm



Growth rate

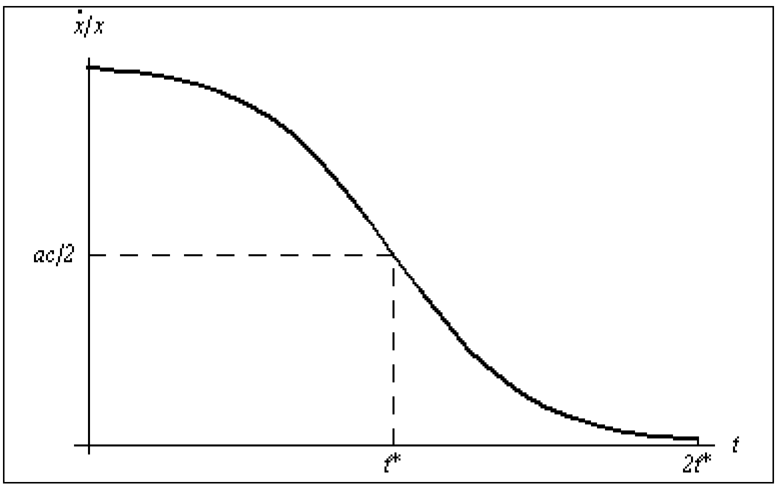
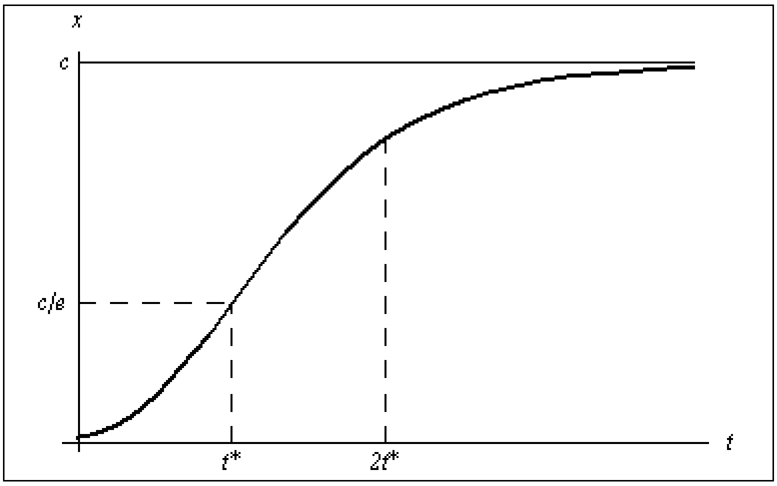


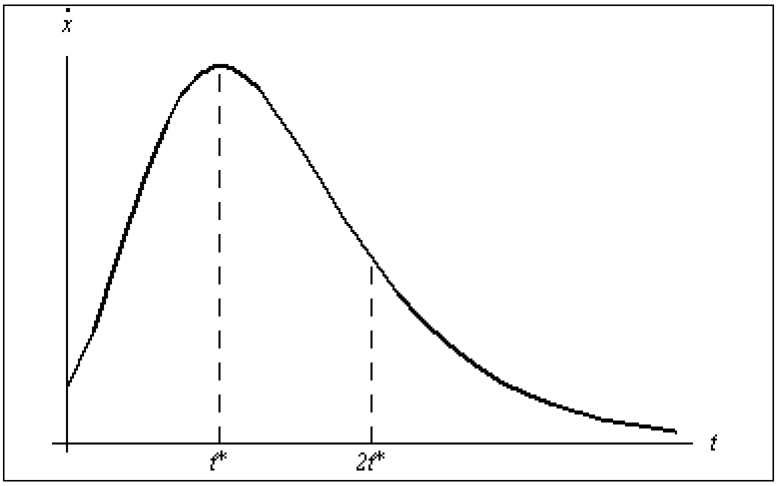
Figure 2: Representations of the conventional Gompertz dynamic

($a = 2$, $c = \sqrt{e}$, $x_0 = 0.02$)

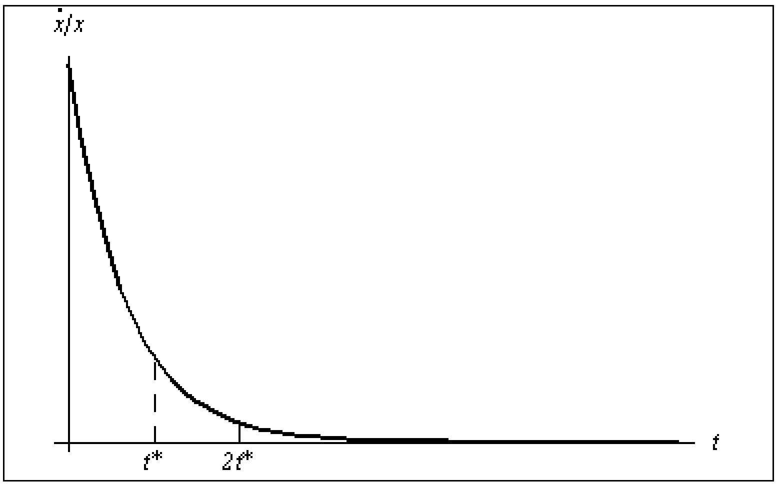
Solution path



Growth rhythm



Growth rate



Moreover, as can be seen in Figure 1, the logistic dynamic is characterized by strong symmetries. Thus, if the turning point of $\dot{x}(t)$ occurs at t^*, $(t^*, c/2)$ is the point of symmetry of $x(t)$, $\dot{x}(t)$ is symmetrical with respect to t^*, while the growth rate $\dot{x}(t)/x(t)$ is symmetrical with respect to $(t^*, ac/2)$, with the period of accelerated growth being equal to the decelerated growth phase. These symmetrical properties again contradict current evidence. Thus, for example, Galvão (2002), Harding and Pagan (2002), Balke and Wynne (1995) and King and Plosser (1994) point out to the existence of expansionary periods that are longer than the ensuing contraction, and where the growth rates involved differ substantially.

In short, we find that it is not possible to define the expansion (depression) period associated with an increasing (or decreasing) growth rate using either the logistic or Gompertz curve, because both show a declining growth rate. In order to overcome this problem, some scholars identify expansion (depression) with acceleration (deceleration), that is to say, with an increasing (or decreasing) growth rhythm. However, this does not resolve another empirical fact, namely that expansionary periods are longer than depressions. In this regard, the very symmetry of the logistic curve means that the period of expansion is equal to that of contraction, while in the Gompertz curve the period of acceleration $[0, t^*]$ is shorter than that of deceleration, as can be seen in Figure 2.

Both the logistic and the Gompertz curves have a floor equal to zero. This creates difficulties in achieving a good fit between the curves and certain highly significant economic and demographic variables such as output, level of technology and population that rarely have an initial value of zero. Naturally, we might consider only increments above a certain value but, in this case, the variable modeled is not exactly the same and may have other properties, especially if we examine the growth rates involved, see Oliver (1982) or recall that, in general, $\frac{\dot{x}(t)}{x(t)} \neq \frac{\dot{\overline{x(t)+k}}}{x(t)+k}$.

The existence of a ceiling also causes problems for the use of both types of curves. Thus, if we know $x(t^*)$, the turning point of $\dot{x}(t)$, then the value of the ceiling will be given as $c = 2x(t^*)$ in the logistic curve and $c = \mathrm{e}\, x(t^*)$ in the Gompertz curve. These relations are too rigid for the empirical data.

In summary, while it is true that the logistic and Gompertz curves work well in theoretical models, their empirical use demands a widening of the spectrum of the sigmoid curves applied. Such a widening is the central objective of this paper, where we first extend the family of S-shaped curves before going to apply them empirically to the *USA Capacity Index* series.

The rest of the paper is organized as follows. Section 2 defines a family of differential equations, which include (1) and (2) and reveal S-shaped dynamics. We also obtain three sub-families of these equations, which admit functional representation and depend on five easily interpreted parameters. In Section 3 we show that any of these sub-families is sufficient to cover the requirements of empirical inquiry. Section 4 is devoted to the empirical application, with our aim being to replicate certain well-known economic phenomena. Section 5 closes the paper with a review of the main conclusions. The Appendix contains the proofs of the propositions, which are also given in detail in Jarne and Sánchez Chóliz (2002).

2. S-SHAPED DYNAMICS

2.1 General description of sigmoid curves

We shall begin by considering a general and increasing S-shaped evolution, that is to say, accelerated growth processes followed by deceleration without descending to negative growth. This is the upward movement from a trough to a peak in the growth process. Let us consider the dynamics $\dot{x} = g(x)$, where $g(x)$ gives the following:

I. $g(x)$ is C^0 over $[f, c]$ and C^1 over (f, c) , $0 \leq f < c$.
II. $g(f) = g(c) = 0$
III. $g(x) > 0$ if $x \in (f, c)$
IV. $g'(x) > 0$ if $x \in (f, x^*)$ and $g'(x) < 0$ if $x \in (x^*, c)$, with $x^* \in (f, c)$,
V. $g(x)$ is C^2 over (f, x^*). Also, $g''(x) > 0$ if $x \in (f, x_1)$ and $g''(x) < 0$ if $x \in (x_1, x^*)$, with $f \leq x_1 < x^*$.

The $g(x)$ that verify these five properties are a sub-family of unimodal curves[1] over $[f, c]$, which we shall henceforth refer to as Ω. Property III ensures that the solution $x(t)$ of $\dot{x} = g(x)$ is increasing, while IV implies accelerating growth between f and x^*, and decelerating or delayed growth between x^* and c. The curve is increasing and, hence, II ensures that f is the floor and c the ceiling for the relevant dynamic. In short, the first four conditions of Ω ensure an S-shaped evolution. Property V is not necessary to obtain this type of evolution, although

[1] Alternative definitions also exist. Thus g: $[f, c] \rightarrow \Re$ is unimodal if I, II, III and IV are verified.

it is a sufficient condition for the growth rate to have only one increasing period and a subsequent decreasing one, as we shall see below.

Clearly, this formulation does not reflect the passage from a peak to a trough, which can be represented exactly through the opposite dynamic, $\dot{x} = -g(x)$. The formal analogy between the two cases is so high that we shall only examine the increasing case, since the conclusions for the decreasing case are evident.

It may easily be shown that the dynamics $\dot{x} = g(x)$ includes the following:

a) $\dot{x} = \sin(x)$ and $\dot{x} = \sin^2(x)$ with $x \in [0, \pi]$;

b) $\dot{x} = \cos(x)$ and $\dot{x} = \cos^2(x)$, with $x \in [-\pi/2, \pi/2]$;

c) $\dot{x} = a\, x\, (c - x)$, where $a > 0$ and $c > 0$, (conventional logistic equation);

d) $\dot{x} = a\, x\, (lnc - lnx)$, where $a > 0$ and $c > 0$ (conventional Gompertz equation).

The properties of the increasing dynamics associated with Ω are reflected in Proposition 1, the proof of which is set out in the Appendix.

Proposition 1. If $g(x) \in \Omega$, the dynamics described by $\dot{x} = g(x)$ have the following characteristics:

1) $x(t)$ increases rapidly (with accelerated growth) until the moment t^* [$x^* = x(t^*)$], and thereafter continues to increase more slowly (with decelerated growth), converging with c, which is an attractor. Thus, the turning point of $\dot{x}$ occurs at t^*.
2) If $x_1 = f = 0$, the growth rate $\gamma(t) = \dfrac{\dot{x}(t)}{x(t)}$ is decreasing at any moment t.
3) If $x_1 > f = 0$ or if $f > 0$, $\gamma(t)$ increases until moment $\hat{t} < t^*$ and decreases for any $t > \hat{t}$. Consequently, $\hat{t}$ is the turning point of the growth rate.

This proposition proves that the dynamics associated with the unimodal curves of Ω provide a good fit with the properties we are looking for, as discussed in the Introduction. The dynamics are S-shaped and the growth rates are increasing and then decreasing if $x_1 > f = 0$, or if $f > 0$. Moreover, the turning points of the rhythm and rate of growth will enable us to distinguish the different phases (emergent, inflexion and saturation) which underlie the processes which follow an S-shaped evolution and which therefore can be modeled with this type of theoretical representation. The proposition is applicable to the logistic and Gompertz equations because both belong to Ω. It can easily be shown that for both $x_1 = f = 0$. Hence, their growth rates are constantly decreasing, as we can see in Figures 1 and 2, which results in problems in many empirical applications. Nevertheless, their dynamics have specific mathematical expressions and it is possible to obtain the expressions for some of their characteristic values. This has encouraged their use for both theoretical and empirical purposes.

Specifically, the solution trajectories, rhythm of growth, growth rate and the values t^* and x^* are given for the logistic equation if $x_0 \in \left(0, \frac{c}{2}\right)$, where:

$$x(t) = \frac{1}{\frac{1}{c} + \left(\frac{1}{x_0} - \frac{1}{c}\right) e^{-act}},$$

$$\dot{x}(t) = \frac{ac\left(\frac{1}{x_0} - \frac{1}{c}\right) e^{-act}}{\left(\frac{1}{c} + \left(\frac{1}{x_0} - \frac{1}{c}\right) e^{-act}\right)^2},$$

$$\gamma(t) = \frac{\dot{x}(t)}{x(t)} = \frac{ac\left(\frac{1}{x_0} - \frac{1}{c}\right) e^{-act}}{\frac{1}{c} + \left(\frac{1}{x_0} - \frac{1}{c}\right) e^{-act}},$$

$$t^* = \frac{1}{ac}\, ln\left(\frac{c}{x_0} - 1\right) \text{ and } x^* = x(t^*) = \frac{c}{2}. \qquad (3)$$

This is also the case for the Gompertz equation if $x_0 \in \left(0, \frac{c}{\mathrm{e}}\right)$, where:

$$x(t) = \mathrm{e}^{\ln c - (\ln(c/x_0))\, \mathrm{e}^{-a\,t}};$$

$$\dot{x}(t) = a \ln(c/x_0)\, \mathrm{e}^{\ln c - at - (\ln(c/x_0))\, \mathrm{e}^{-a\,t}};$$

$$\gamma(t) = \frac{\dot{x}(t)}{x(t)} = a \ln(c/x_0) \mathrm{e}^{-at},$$

$$t^* = \frac{1}{a} \ln(\ln(c/x_0)) \text{ and } x^* = x(t^*) = c/\mathrm{e}. \qquad (4)$$

2.2 Some relevant sub-families of sigmoids

The dynamics defined by Ω have one great advantage, namely that they cover practically the whole spectrum of increasing S-shaped curves, however, they have two notorious disadvantages. Firstly, Ω is a function set with infinite dimension, according to the great variability associated with the emergent processes of growth and self-organization. On the other hand, the many Ω-functions have not an easy mathematical representation. In view of this, we have selected three sub-families that can be characterized as providing good cover of the spectrum of increasing sigmoid curves from the empirical point of view, depending on five easily understandable parameters and being mathematically manageable. In other words, each of these subfamilies reduces the key properties of any sigmoid evolution to only five parameters. The first of these is given by:

$$\dot{x} = a\,(x - f)^{b}\,(c - x)^{d}, \text{ where } a, b, d > 0, \text{ and } c > f \geq 0. \tag{5}$$

and where we may refer to its members as *generalized logistic equations*. The second is:

$$\dot{x} = a(x-f)^{b}\left(\ln\frac{c-f}{x-f}\right)^{d} \text{ if } x > f,\ \dot{x} = 0 \text{ if } x = f, \text{ where } a, b, d > 0, \text{ and } c > f \geq 0, \tag{6}$$

We shall call each of these *generalized Gompertz equations*. The third is described as follows:

$$\dot{x} = a\,(x - f)^{b}(e^{c} - e^{x})^{d}, \text{ where } a, b, d > 0, \text{ and } c > f \geq 0, \tag{7}$$

and we shall call its members *exponential sigmoid equations.*

Note that if in (5) and (6) $b = d = 1$ and $f = 0$, then we obtain the conventional logistic and Gompertz equations, respectively.

As stated above, the curves depend on five parameters, a, f, c, b, and d in each of the three sub-families. The parameter a is a structural indicator of the speed of change of the variable and does not depend on the evolution of the dynamic. It is a typical coefficient in conventional logistic or Gompertz equations, as can be seen in (1) and (2). In the context of technological change, development and the emergence of innovations in a sector depend on both their purely technological characteristics and the growth of business. The former is reflected by a, while f, c, b and d reflect the latter.

The parameter f represents the floor, which may be positive, and the parameter $c > f$ is the ceiling. The value c-f is the available range for the variable and is similar to c from (1) and (2), where f is equal to 0. However, the existence of two parameters, c and f instead of c-f, increases the information within the dynamic equation. Their meanings will both depend on the variable represented and the analytical schedule. For example, in the diffusion of technological revolutions or innovation packages, the ceiling is the maximum technological level attainable in the present techno-economic paradigm, whilst the floor represents the minimum level, which would already have been reached in the preceding paradigm. Thus, c-f is the size of the niche or capacity limit. The starting point is close to f. The instability typical of the saturation phase is associated with c and not f.

In these three sub-families, the gap to the floor affects the dynamic via the term $(x\text{-}f)^b$, which represents the self-reinforcing effect. This might be called the floor effect and it is the relevant effect during the emergent phase. Similarly, as the dynamic approaches the ceiling, a braking or exhaustion effect appears, which we shall call the ceiling effect. This is the characteristic effect of the saturation phase. In the three sub-families defined, this effect is captured via the terms $(c\text{-}x)^d$, $\left(\ln\frac{c\text{-}f}{x\text{-}f}\right)^d$ and $(e^c\text{-}e^x)^d$, respectively. In (5), (6) and (7), the two effects are multiplicatory, in order to incorporate the endogenous linkages of the underlying self-organization process more effectively. It is important also to note that the exponents *b* and *d* make the relationship captured between growth and increase in the variable much more complex than in conventional equations, improving the explanatory ability and adaptability of the two effects.

The parameters *b* and *d* introduce asymmetry into the dynamics and measure the greater or lesser relevance of the floor or ceiling gap. This is another way to capture the endogenous complexity of the process. The key to calculating asymmetry and the relative weight of each of these factors is the ratio $\frac{b}{d}$, as will be seen in Proposition 4. Thus, the influence of the floor is greater the higher the ratio (see Figures 3, 4 and 5). The growth rate in the generalized logistic curves takes the form of a bell biased toward the right when $\frac{b}{d} > 1$, and towards the left when $\frac{b}{d} < 1$. Hence, in these logistic equations the expansionary phase lasts longer than the contraction phase whenever $\frac{b}{d} > 1$.

Later we will see in the applications how the use of the five parameters enables us to obtain good estimates which are, moreover, better than those obtained with conventional logistics equations and with the Gompertz function. Furthermore, if we adjust two successive sigmoid functions in one same series, as we will do in the applications, the changes and parametric values will give us the structural change which appears after the period of instability and uncertainty in the saturation phase.

Let us now show that the dynamics defined by (5), (6) and (7) fulfill the properties of Proposition 1.

Proposition 2. The dynamics described by (5), (6) and (7) validate the following:

1) They are specific cases of $\dot{x} = g(x)$, with $g(x)\in\Omega$ and, consequently, they have the properties already seen in Proposition 1. Moreover, $x_1 > f \Leftrightarrow b > 1$ in each of them.

2) The growth rate $\gamma(t) = \dfrac{\dot{x}(t)}{x(t)}$ is decreasing if $f = 0$ and $b \leq 1$. In the remaining cases $\gamma(t)$ increases up to $\hat{t} < t^*$, in which $x(\hat{t}) = \hat{x}$, and thereafter decreases.

3) In the generalized logistic equations, we have that:

   a) $x^* = \dfrac{bc+df}{b+d}$;

   b) if $b > 1$ and $f = 0$, $\hat{x} = \dfrac{c(b\text{-}1)}{b+d\text{-}1}$;

   c) if $b+d = 1$ and $f > 0$, $\hat{x} = \dfrac{fc}{cd+fb}$;

   if $b+d \neq 1$ and $f > 0$, $\hat{x} = \dfrac{bc+df\text{-}f\text{-}c+\sqrt{(bc+df\text{-}f\text{-}c)^2+4fc(b+d\text{-}1)}}{2(b+d\text{-}1)}$.

4) In the generalized Gompertz equations we have that:

   a) $x^* = f + \dfrac{c\text{-}f}{e^{d/b}}$;

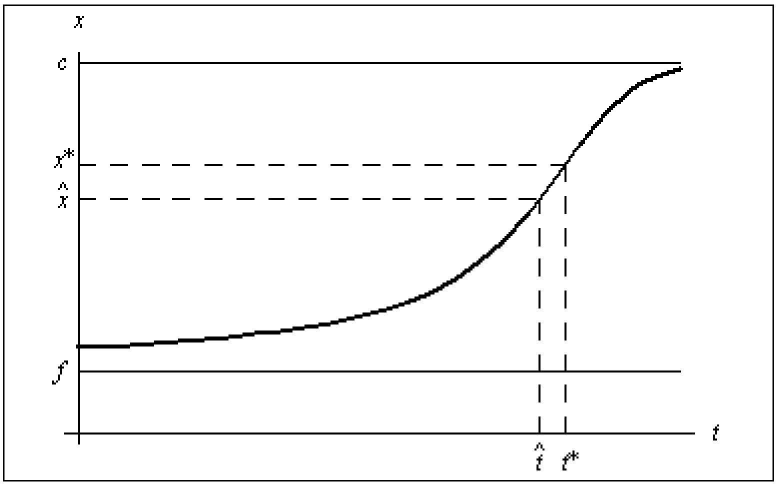
   b) if $b > 1$ and $f = 0$, $\hat{x} = c\, e^{-d/(b-1)}$.

Figures 3, 4 and 5 show the dynamics of a equation for each sub-family, all with a positive floor and $b > d$. Let us recall that t^* and $\hat{t}$, which appear in these figures, are the moments in time when the turning points of $\dot{x}(t)$ and $\dot{x}(t)/x(t)$ occur respectively. From these points on, the limits of the different phases (emergent, inflexion and saturation) of the sigmoid evolution may be defined. In our analysis we shall assume that the emergent phase corresponds to the time interval $[0, \hat{t}\text{-}\varepsilon)$ with $\varepsilon = t^*\text{-}\hat{t}$; the inflexion phase to $[\hat{t}\text{-}\varepsilon, \hat{t}+\varepsilon = t^*]$ and the saturation phase to $(t^*,\infty]$. Thus, in the emergent phase both the rate and rhythm of growth are increasing and the growth is strongly accelerated. Inversely, in the saturation phase the rate and rhythm of growth are decreasing and the growth is strongly decelerated. Finally, the inflexion phase has the turning point of the growth rate as its central point, and the growth rhythm is increasing but is approaching its maximum. Growth is therefore approximately linear.

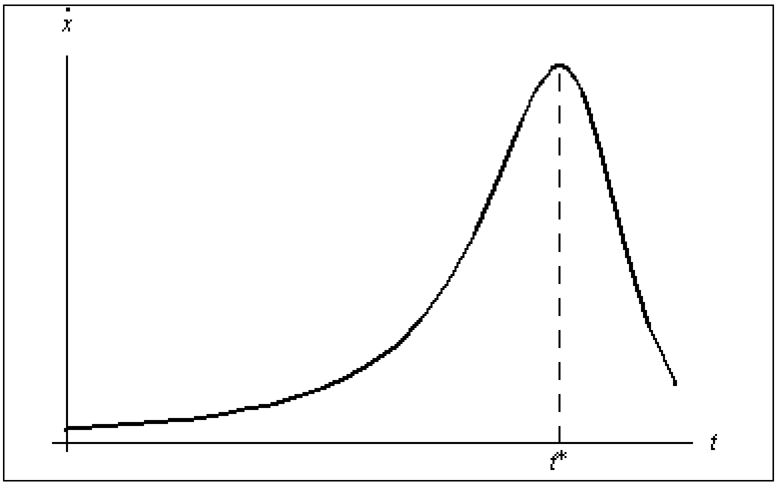
Figure 3: Representations of a generalized logistic dynamic

($a = 2, f = 0.1, c = 0.6, b = 2, d = 1, x_0 = 0.138$)

Solution path



Growth rhythm



Growth rate

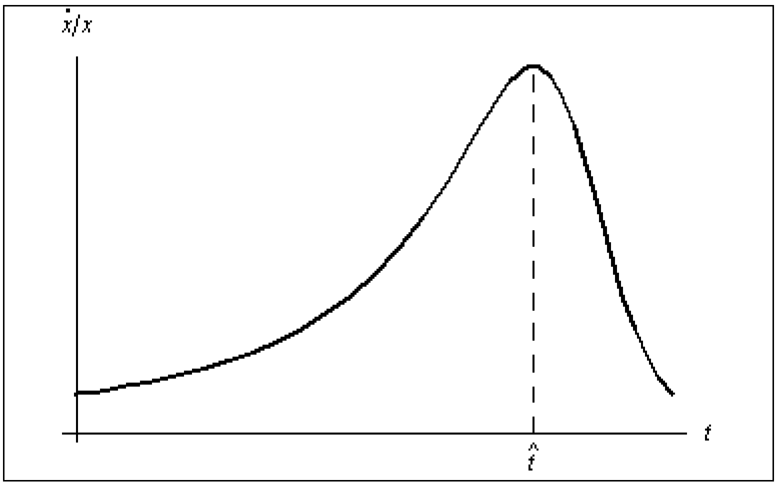
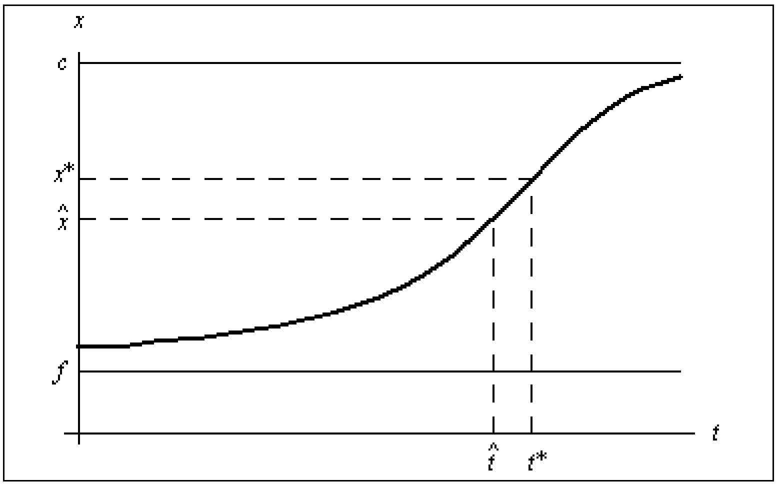


Figure 4: Representations of a generalized Gompertz dynamic

($a = 2, f = 0.1, c = 0.6, b = 2, d = 1, x_0 = 0.138$)

Solution path



Growth rate

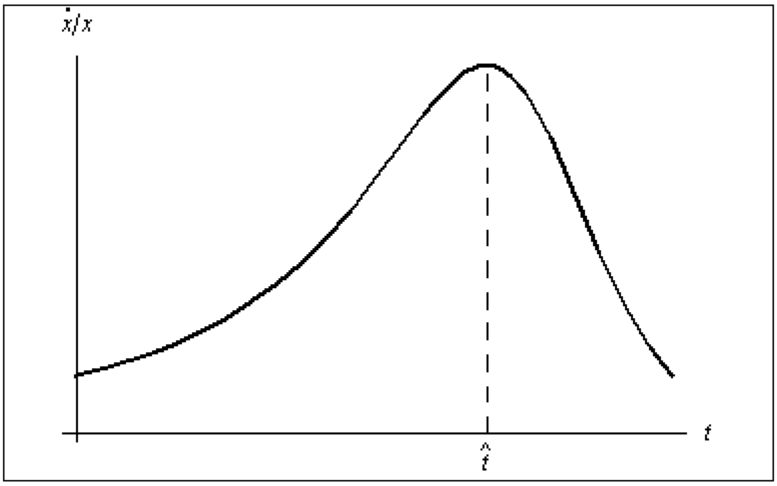
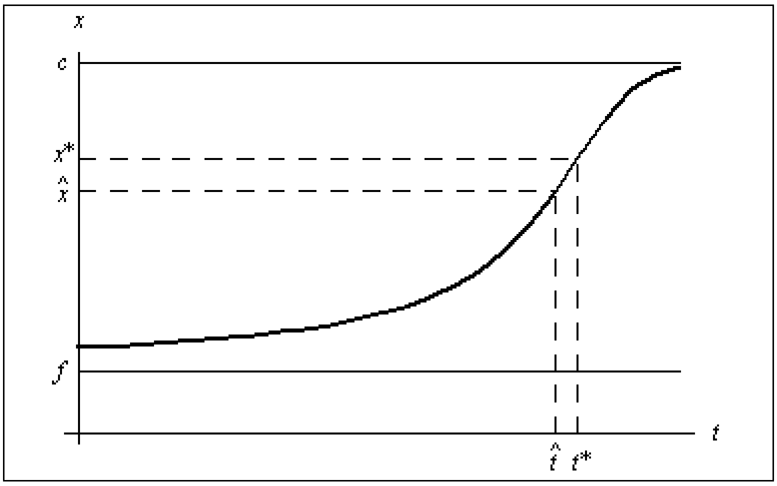


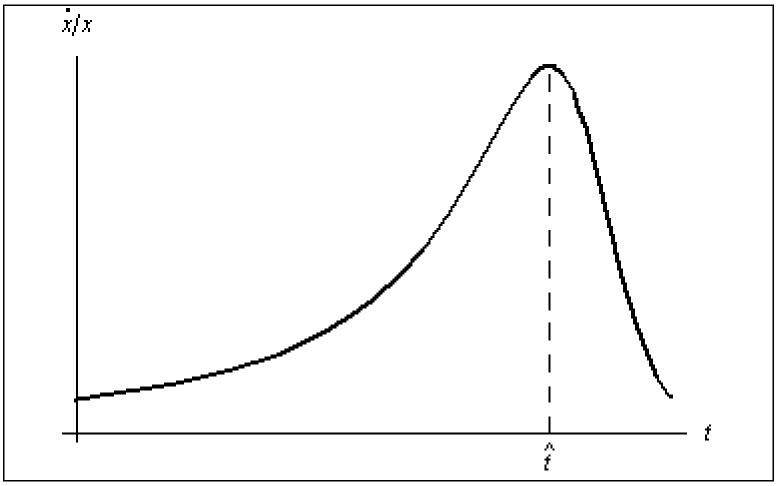
Figure 5: Representations of an exponential sigmoid dynamic

($a = 2, f = 0.1, c = 0.6, b = 2, d = 1, x_0 = 0.138$)

Solution path



Growth rate



3. PROPERTIES OF THE SUB-FAMILIES

As can be seen in Proposition 1, the dynamics associated with Ω take certain characteristic values reflecting the greater part of the economic information, as follows:

1) The floor f of the dynamic.
2) The ceiling c of the dynamic.
3) The value x^* where the rhythm of growth reaches its maximum.
4) The value $\hat{x}$ where the growth rate reaches its maximum.

5) A development time T, which is usually identified through $\hat{t}$, that is to say, the moment at which the growth rate turning point appears, or through t^*, the growth rhythm turning point, or through the time period required to pass from an initial value of $x_0 = f+\varepsilon$ to a final value of $x_F = c-\varepsilon$, which we shall call T_F.

In Proposition 3, we see that there is always a generalized logistic equation, a generalized Gompertz equation and an exponential sigmoid equation that coincide with the five values which characterize any sigmoid curve.

Proposition 3. Let f_d, c_d, x_d^*, $\hat{x}_d$ and T_d be given values, such that $0 \leq f_d \leq \hat{x}_d < x_d^* < c_d$. Thus,

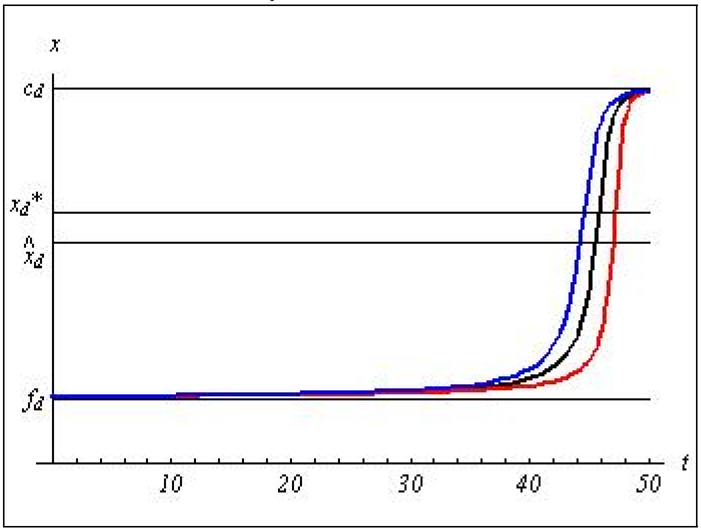
1) If $f_d < \hat{x}_d$ and $T_d > 0$, a generalized logistic equation, a generalized Gompertz equation and an exponential sigmoid equation exist, validating $f = f_d$, $c = c_d$, $x^* = x_d^*$, $\hat{x} = \hat{x}_d$ and $\hat{t} = T_d$.
2) If $f_d \leq \hat{x}_d$ and $T_d > 0$, point 1 is verified where $t^* = T_d$ instead of $\hat{t} = T_d$.
3) If $f_d \leq \hat{x}_d$ y $T_d > 0$, point 1 is verified where $T_F = T_d$ instead of $\hat{t} = T_d$.

The significance of this proposition is clear. If we require a sigmoid evolution to maintain only five given values, then its dynamics can be represented with elements from any of the three sub-families without causing relevant errors. Consequently, almost all S-shaped evolutions can be adjusted using elements from any of the three sub-families, since the five values capture the most relevant empirical information and we may ignore other kinds of sigmoid curves. A graphic proof of this result is shown in Figures 6 and 7. By way of example, Figure 6 represents the results of adjusting a logistic, Gompertz and exponential sigmoid dynamic for an S-shaped evolution defined by: $f_d = 0.1$, $c_d = 0.6$, $\hat{x}_d = 0.35$, $x_d^* = 0.4$, $\varepsilon = 0.005$. In the first case, the duration of the process is adjusted by $T_F = 50$, in the second by $t^* = 30$, and in the third by $\hat{t} = 25$. Figure 7 shows the fit of an actual series, (*USA Capacity Index for Total Industry*)/10, with the three types of curve, which yet again confirms the result given in Proposition 3.

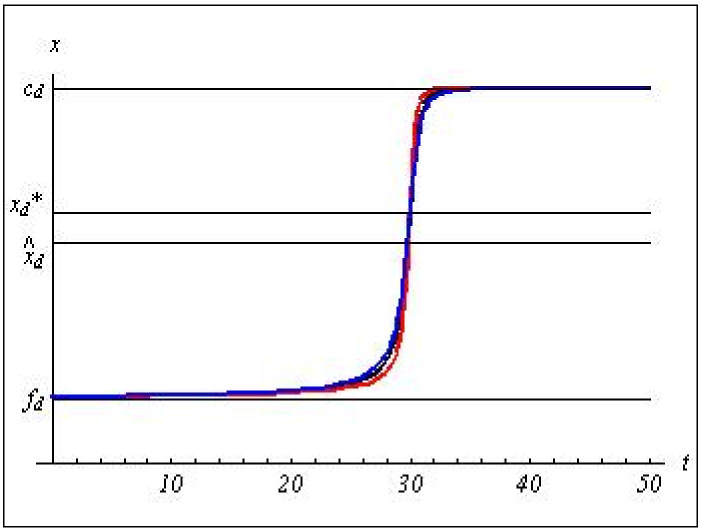
Figure 6: Fits for $f_d = 0.1$, $c_d = 0.6$, $\hat{x}_d = 0.35$, $x_d^* = 0.4$, $x_0 = 0.105$

(_____ generalized logistic curve; _____ generalized Gompertz curve; _____ exponential sigmoid curve)

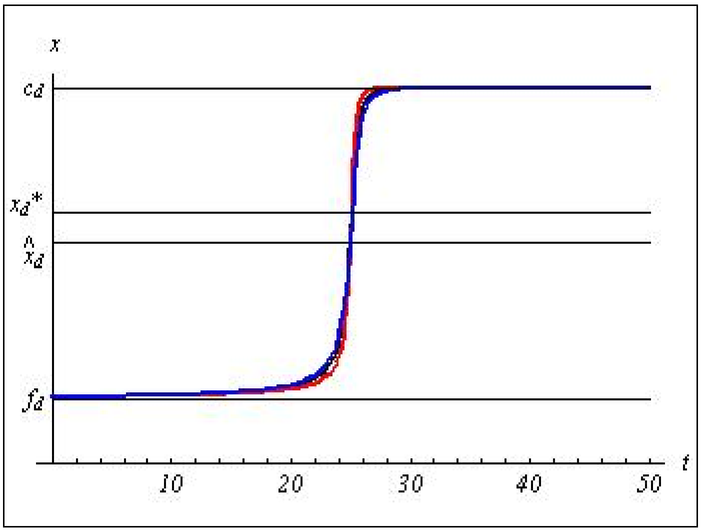
Case 1: $T_F = 50$ for the three curves



Case 2: $t^* = 30$ for the three curves



Case 3: $\hat{t} = 25$ for the three curves



On the basis of Proposition 3, any of the three sub-families may be used for the purposes of empirical research, although it would be reasonable to consider whether any of the three provides particular advantages. In our opinion, there are good reasons to opt for the generalized logistic curves. The interpretation of their parameters is less demanding because the expression of the ceiling effect is simpler and, as we saw in Proposition 2, the calculation of the characteristic values is easy. This provides a better understanding of the economic realities, while facilitating empirical investigation. Certain other properties of this sub-family can be seen in Proposition 4:

Proposition 4. The set of dynamics governed by (5) validate:

1) $\frac{b}{d} = \frac{x^*-f}{c-x^*}$. Hence the point x^* divides the gap between the ceiling c and the floor f into two segments that are proportional to b and d.
2) If two dynamics coincide at a, f, c, $\mu = b_1/d_1 = b_2/d_2$ and $b_2 = \lambda b_1$, then
   a) for any x of (f, c), $\dot{x}_2(x) = a\,[h(x)]^{\lambda}$, where $\dot{x}_1(x) = ah(x)$,
   b) $h(x^*) = \left(\frac{c-f}{b_1+d_1}\right)^{b_1+d_1} b_1^{\,b_1}\, d_1^{\,d_1}$
   c) if $\frac{\dot{x}_1(x^*)}{a} = h(x^*) > 1$ ($\Leftrightarrow \frac{\dot{x}_2(x^*)}{a} = [h(x^*)]^{\lambda} > 1$), the dynamic with the highest values for b and d is faster in the central area in an open U containing x^*, and is slower in the area (f, c)-U.

The significance of this proposition is clear. Point 1 provides information on the areas presenting accelerated and decelerated growth. Since x^* separates these areas, the proposition also tells us that the area of accelerated movement is greater than, equal to or less than the area of decelerated movement if and only if $\frac{b}{d}$ is >, =, < 1. In other words, the greater b in relative terms, the more intense will be the floor acceleration effect and the less so the ceiling braking effect. In a way, then, b and d are measures of the endogenous impetus / repulsion provided by the floor / ceiling.

We know that parameter a is a structural indicator of the growth rhythm and that the growth is also affected by c, f, b and d. Points 2-a and 2-b shed some light on the relationship between the parameters and the growth rhythm. Finally, if $h(x^*) > 1$ the point 2-c confirms that the greater the values of b and d are, the greater the growth rhythm in the central band between the floor and ceiling is, and the lower it is in the proximity of the floor and ceiling.

4. EMPIRICAL APPLICATION TO TIME SERIES

4.1 Fit with the *USA Capacity Index for Total Industry* (*USA CI*) from February 1986 through January 2003

One of the objectives of researchers who have worked with sigmoid curves has been to obtain dynamic representations providing a good fit with series describing technological and demographic change, in the hope that these fits would improve the forecasting techniques. However, success has only been relative. While it is true that we should not expect results for all time series, it is nevertheless possible to obtain a good fit for certain significant series, as we shall see, and this makes it possible to extract relevant information.

Figure 7 presents the fit in three cases, each using one of the three sub-families defined above, to the *USA Capacity Index for Total Industry* from February 1986 through January 2003 (Source: Ecowin)[2]. Each of the three curves provides a good visual approximation to the series and, indeed, it is difficult to say which is best. This confirms Proposition 3.

Table 1 presents the results for seven fits: the three earlier fits obtained with the generalized equations and the last four obtained using standard logistic and Gompertz equations; two with a zero floor and two with a positive floor[3]. The econometric indicators prove that the fits obtained with the generalized equations, the first three, are better than those obtained with the standard equations – the last four – thus confirming that the equations proposed are better. In addition, the fits associated with each of the generalized sigmoid curves have very similar reliability. The values which each of them assign to the floor and ceiling, f and c, differ very little. The values of a, b and d are different because they do not represent the same thing in each curve. However, it can be verified that they provide the same qualitative information.

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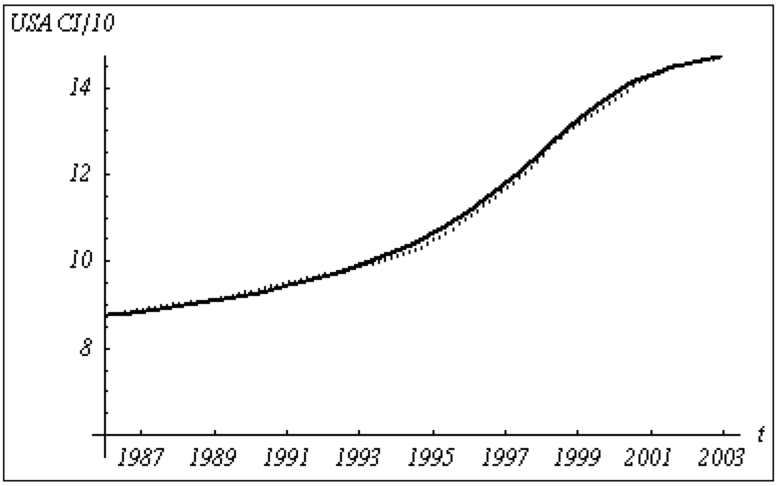
[2] The series divided by 10 and not the original one is considered due to overflow problems because of $e^c - e^x$ when fitting the exponential equation.

[3] Equations were estimated by non-linear least-squares.

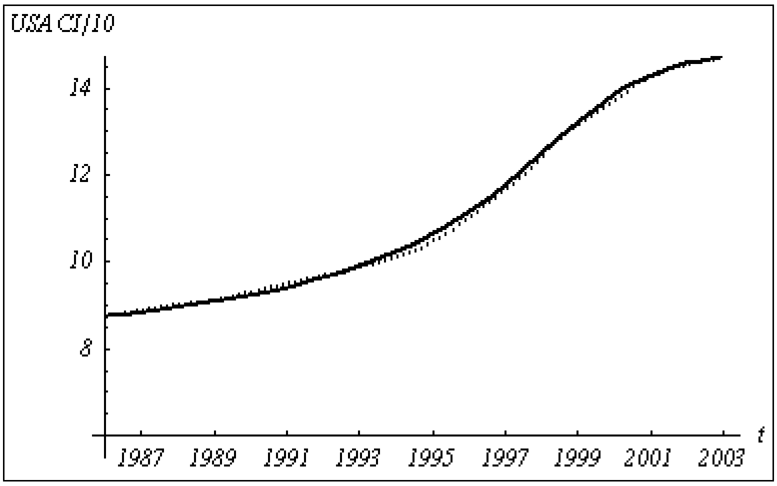
Figure 7: Fits to the *USA Capacity Index* for Total Industry series

(······ real series; —— fit)

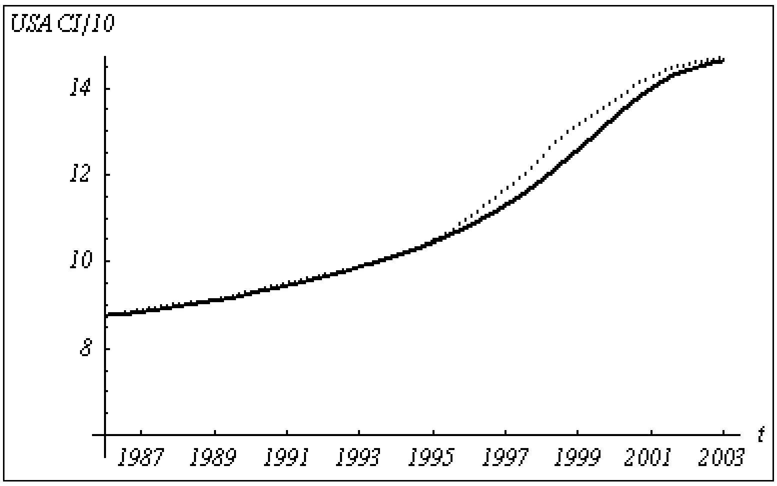
Fit with generalized logistic equation



Fit with generalized Gompertz equation



Fit with exponential sigmoid equation



If we consider the generalized logistic equation, we find that *b/d* = 2.58 > 1, which confirms that the accelerated growth phase is longer than (in fact, more than twice as long as) the decelerated growth phase. In other words, the dynamics presents asymmetry which cannot be captured by a standard logistic. We also find that both *b* and *d* are greater than 1, which means that the growth rhythm is greater than would be the case using a standard logistic equation with a similar *a* coefficient and the same floor and ceiling.

Table 1: Coefficients of fit to the *USA Capacity Index* for Total Industry series

	Generalized logistic	Generalized Gompertz	Exponential sigmoid	Logistic $f = 0$	Gompertz $f = 0$	Logistic $f > 0$	Gompertz $f > 0$
a	1.46×10^{-4}	5.59×10^{-4}	2.68×10^{-14}	9.76×10^{-7}	1.14×10^{-4}	2.14×10^{-3}	8.59×10^{-3}
f	6.80	6.80	6.80	0	0	8.16	8.57
c	14.89	14.89	14.90	2775.47	2.23×10^{11}	17.35	22.38
b	2.91	3.37	1.97	1	1	1	1
d	1.13	1.11	1.69	1	1	1	1
R^2	0.91	0.91	0.85	0.20	0.19	0.61	0.51

4.2 Fit with the *USA Capacity Index* series from February 1967 through January 2003

In this section we shall present the fit, with the *USA Capacity Index* (*USA CI*) series for Total Industry, as well as for various sectors, specifically, *Durable Goods, Manufacturing, Computers & Communication Equipment & Semiconductors* (from now on, *Computers*) and *Primary Processing*, (source: Ecowin). These monthly series are good indicators for capital accumulation and investment processes, a key reason for their selection. The time period considered now is somewhat longer than in the previous fit, running from February 1967 through January 2003, allowing us to capture both the end of the growth period that followed World War II and growth after the crisis of the 1970s.

The results of the fits of different generalized logistic equations are reported in Table 2, which includes the parameter estimates, their *t*-values and R^2. Moreover, it also shows some results obtained from the analysis of the fits.

Table 2: Results for the adjusted logistic equations of the *USA Capacity Index* series (February 1967 - January 2003)

	Sub-period	*a* (*)	*f*	*c*	*b*	*d*	R^2
Total industry	1º	7.19×10^{-2} (-2.45)	49.99 (0.17)	92.00 (10.44)	6.50×10^{-4} (**) (0.01)	2.76×10^{-1} (**) (2.34)	0.26
	2º	2.87×10^{-8} (-3.47)	69.19 (9.18)	150.93 (83.25)	3.07 (3.50)	1.40 (4.32)	0.91
Durable Goods	1º	5.60×10^{-2} (-0.33)	10.01 (0.04)	75.00 (0.06)	2.64×10^{-1} (**) (0.29)	3.26×10^{-5} (**) (7.12×10^{-4})	0.04
	2º	1.72×10^{-5} (-5.89)	68.46 (24.29)	181.85 (61.15)	1.62 (6.87)	1.11 (5.38)	0.92
Manufac-turing	1º	1.18×10^{-1} (-0.11)	5.00 (3.11×10^{-4})	85.87 (13.08)	1.08×10^{-2} (**) (-5.69×10^{-3})	1.13×10^{-1} (**) (-0.77)	0.08
	2º	3.79×10^{-8} (-3.89)	60.00 (6.43)	153.29 (174.57)	3.14 (3.88)	1.09 (6.09)	0.91
Computers	1º	1.41×10^{-4} (-0.57)	0.35 (0.85)	30.00 (0.66)	1.29 (3.23)	1.35 (0.35)	0.95
	2º	2.42×10^{-4} (-8.74)	9.98 (2.15)	496.02 (100.74)	1.23 (12.21)	6.42×10^{-1} (8.38)	0.93
Primary Processing	1º	1.73×10^{-2} (-0.19)	29.53 (0.04)	83.89 (6.35)	9.65×10^{-2} (0.02)	6.43×10^{-1} (0.44)	0.51
	2º	2.85×10^{-10} (-2.54)	55.17 (4.27)	158.51 (45.98)	3.69 (2.61)	1.74 (3.19)	0.89

	Sub-period	*c-f*	*b+d*	*b/d*	$\hat{t}$	*t**	***Emergent phase*** (months from Feb. 1967)	***Inflexion phase*** (months)	***Saturation phase*** (months up to Jan. 2003)
Total industry	1º	42.01	2.76×10^{-1}	2.36×10^{-3}	Decreasing rate	Decreasing rhythm	0	0	227
	2º	81.74	4.47	2.20	January 98	May 98	139	9	56
Durable Goods	1º	64.99	2.64×10^{-1}	8100.7	Decreasing rate	Increasing rhythm	0	227	0
	2º	113.39	2.74	1.46	January 98	October 98	134	19	51
Manufac-turing	1º	80.87	1.24×10^{-1}	9.53×10^{-2}	Decreasing rate	Decreasing rhythm	0	0	227
	2º	93.29	4.23	2.87	January 98	June 98	138	11	55
Computers	1º	29.65	2.65	0.95	May 81	Increasing rhythm	116	111(***)	0
	2º	486.04	1.87	1.91	July 98	September 2000	123	53	28
Primary Processing	1º	54.36	7.40×10^{-1}	1.50×10^{-1}	Decreasing rate	Decreasing rhythm	0	0	227
	2º	103.34	5.42	2.13	January 98	June 98	138	11	55

(*) Parameter α is adjusted instead of *a*, *i.e.* $a = e^{\alpha}$. The *t*-value is given for the fit of parameter α.

(**) The equation fits with $b = \beta^2$ y $d = \delta^2$. The *t*-value is given for the adjustment of parameter β and δ.

(***) Figure 9 shows that $\dot{x}(t)$ is practically at its turning point, *t**, in January 2003.

If we consider a period covering more than one cycle, even if we do not go into the economic and physical fundamentals of the cyclical behavior[4], one would expect the logistic equations for each cycle to have different coefficients reflecting disparities in the technological and productive conditions in the sector. These changes in the coefficients would be the consequence of the structural changes which occur after the period of instability of a saturation phase and their presence would confirm the existence of two cycles.

An initial visual analysis of the evolution of these series over time (see Figure 8) reveals two distinct sub-periods with a structural break, namely before and after 1986. Let us assume that the change in the cycle occurred in January 1986, which saw an (admittedly small) slide in the *USA CI* for Total Industry series, and falls in *Durable Goods*, *Manufacturing* and *Computers*. The other sector examined, *Primary Processing*, had already dipped from July 1982 through October 1983. In any event, a clear change in the pattern of capital accumulation may be discerned in U.S. industry in the first half of the 1980s, (see, e.g. Gualerzi, 2001).

This structural break is also confirmed in Table 2. It may be seen that the a coefficients change radically from the first to the second subperiod, the changes being less in *Computers*. Also, for b and d we observe major changes: ratio b/d increases on passing on to the second subperiod, except in the case of *Durable goods* and the same is the case with the range of the period, c-f.

Table 2 also shows the value t^* at which growth switches from an accelerated to a decelerated rhythm and the moment $\hat{t}$, which is the turning point of the growth rate. Following the criteria indicated in Section 2, we used these time values to fix the limits of the different cycle phases. Let us assume that the emergent phase corresponds to the time interval $[0, \hat{t}-\varepsilon)$ with $\varepsilon = t^*-\hat{t}$, the inflexion phase to $[\hat{t}-\varepsilon, \hat{t}+\varepsilon = t^*]$ and the saturation phase to $(t^*,\infty]$.

Figure 8 provides a graphic representation not only of the series themselves, but also of the fits obtained for each of the sub-periods. Note that the series and fits are very close. Figure 9 presents the corresponding growth rhythms for each series, which may be equated with industrial investment, while Figure 10 shows the growth rates.

With the help of these figures and of Table 2, in the second sub-period, from 1986 to 2003, we can appreciate a clear S-shaped evolution and the presence of the three phases in all cases:

[4] The existence and justification of long-term cycles in economic growth has given rise to so many valuable contributions that it is impossible to quote them all. An approach to long-term cycles may be made through Kuznets (1930), Schumpeter (1939), Goodwin (1967), Van Duijn (1985), Mandel (1995), Freeman (1996), Kwasnicki and Kwasnicka (1996) and Aoki and Yoshikawa (2002)

emergent, inflexion and saturation. It is in *Computers* where the saturation phase is seen later, in September 2000.

Figure 8: *USA Capacity Index* series Adjustments

(•••••• real series; —— fit)

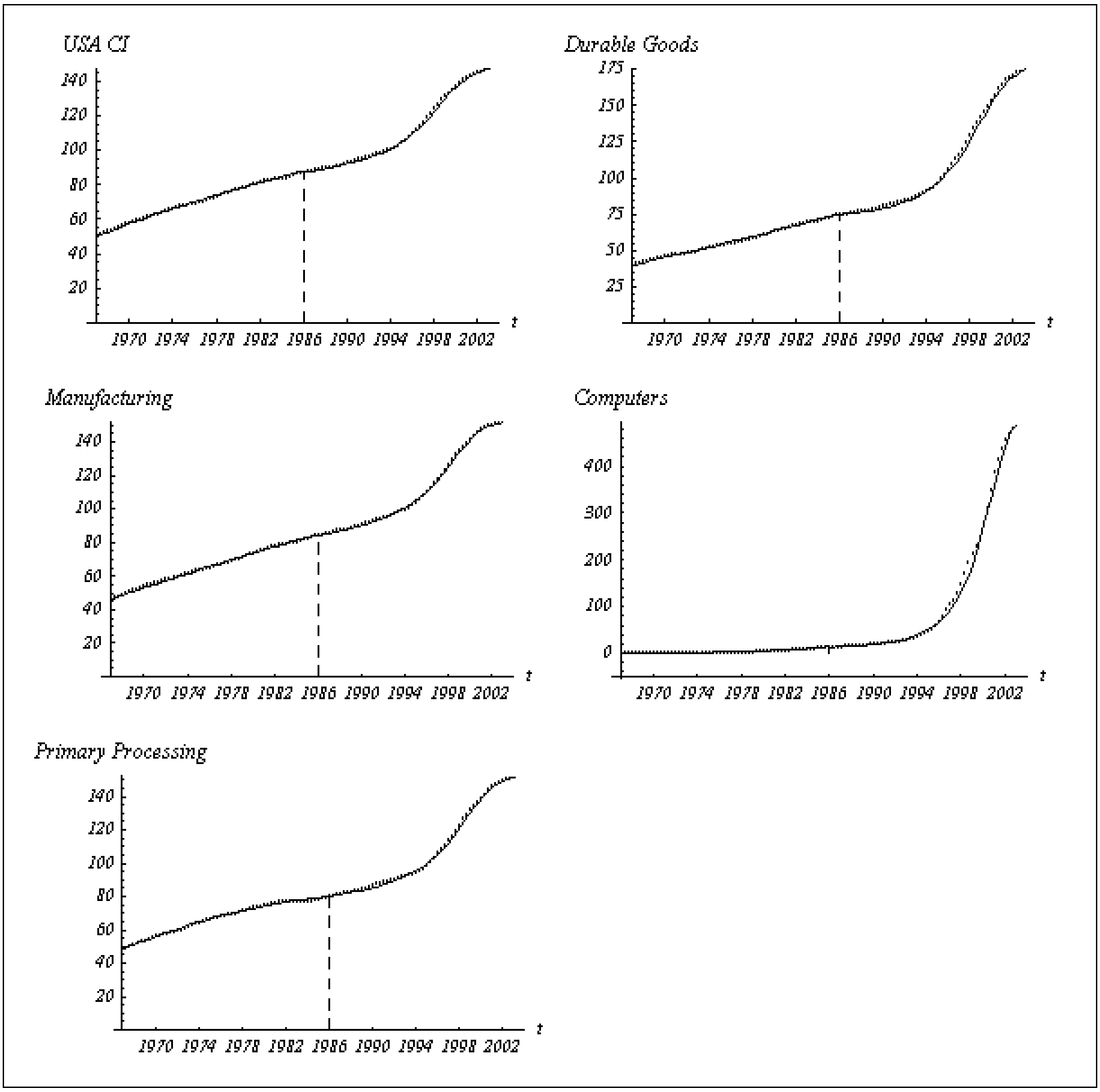
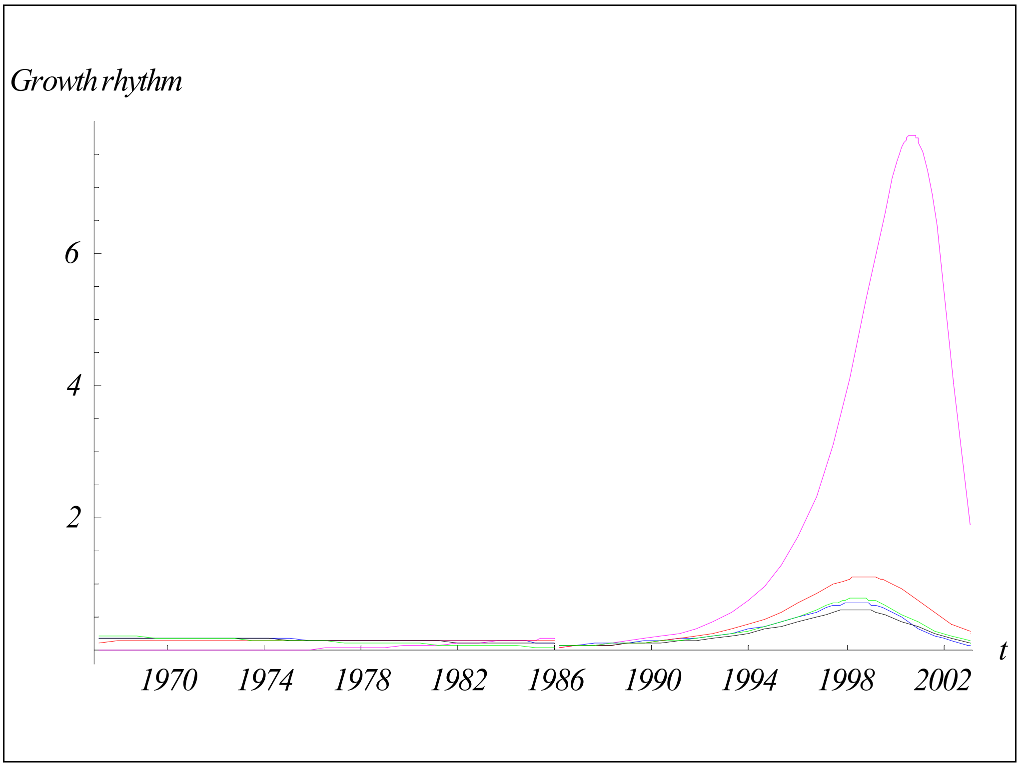


Figure 9: Growth Rhythms in the *USA Capacity Index* series

_____ : TOTAL INDUSTRY; _____ : DURABLE GOODS; _____ : MANUFACTURING;
_____ : COMPUTERS; _____ : PRIMARY PROCESSING



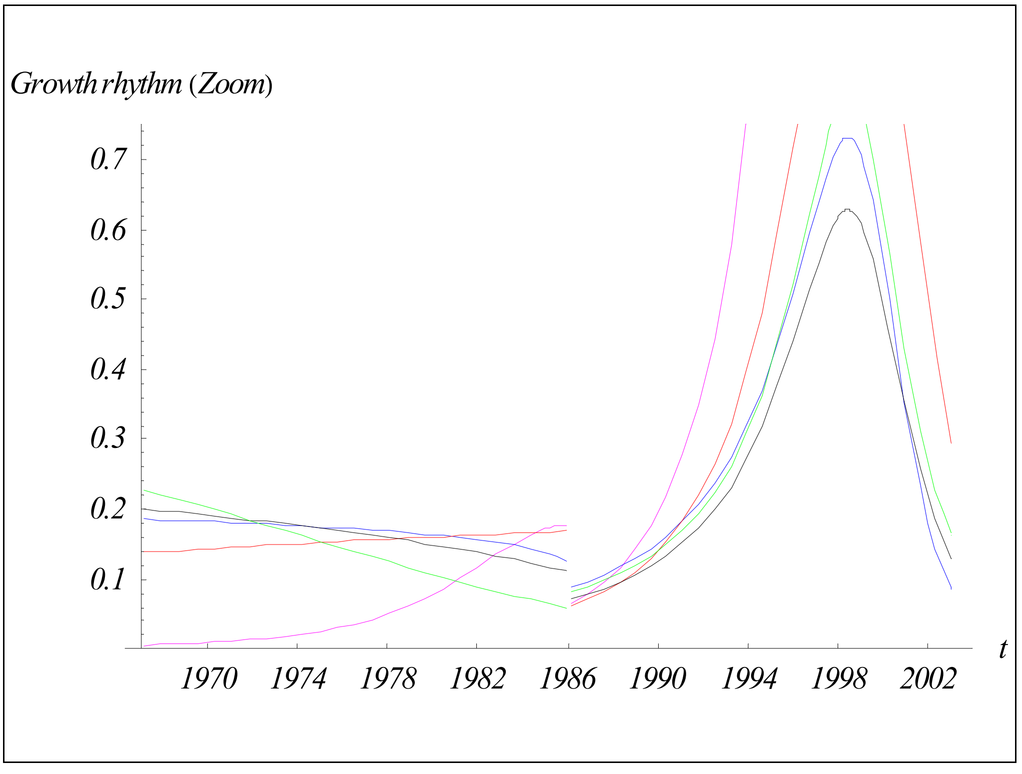
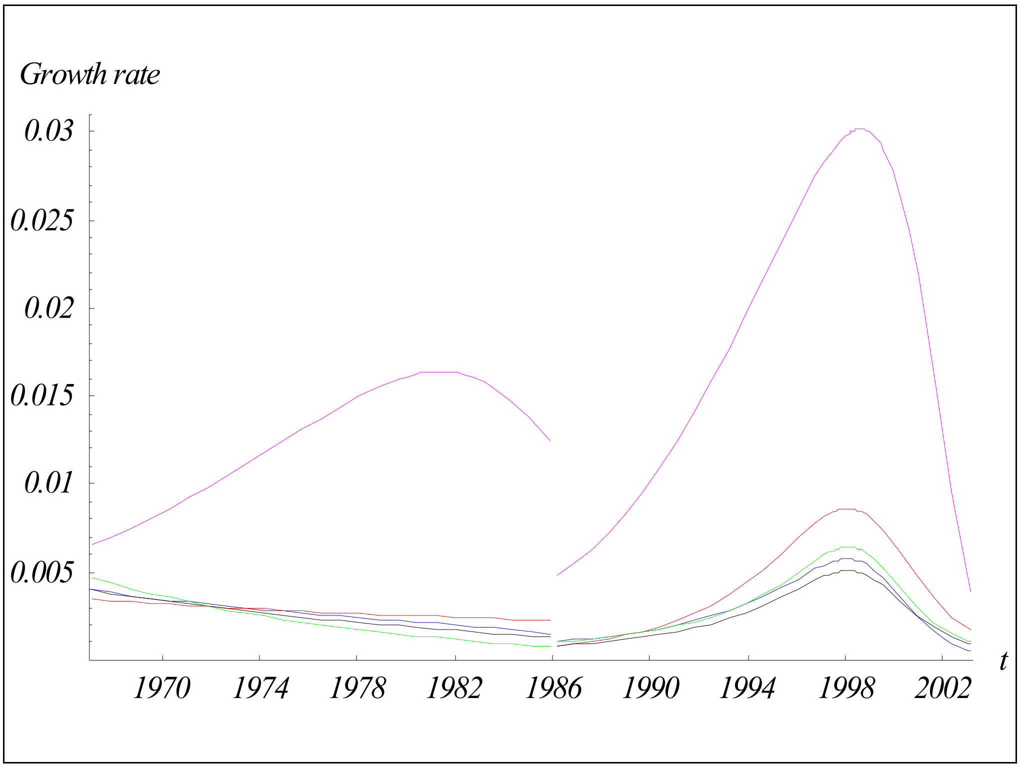
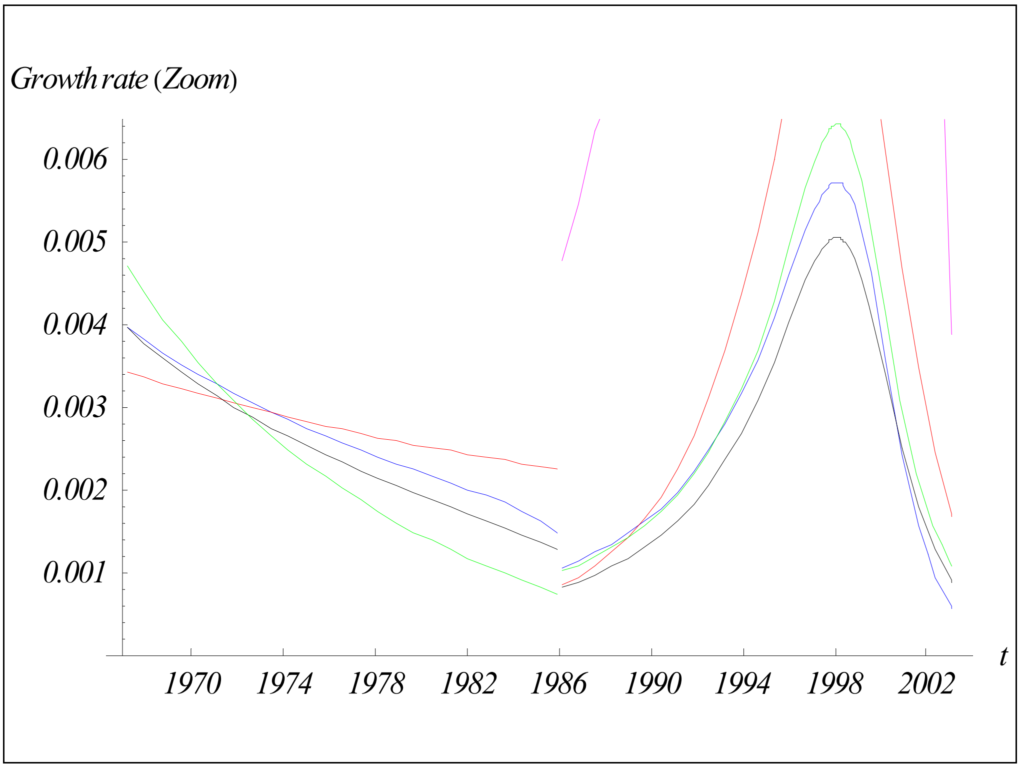


Figure 10: Growth Rates in the *USA Capacity Index* series

______ : TOTAL INDUSTRY; ______ : DURABLE GOODS; ______ : MANUFACTURING;
______ : COMPUTERS; ______ : PRIMARY PROCESSING





By contrast, in the first of the sub-periods, from 1967 to 1986, an S-shaped evolution is obtained, however it is incomplete, in that it does not present the three phases[5]. According to Table 2, *Total Industry* and the sectors *Manufacturing* and *Primary Processing* present only the saturation phase, the sector *Durable Goods* presents the inflexion phase, and in the case of *Computers*, the growth corresponds to the emergent and inflexion phases. The turning point in their rate of growth occurs in May 1981. These results reflects the sharp deceleration of growth that took place in the 1970s due to the energy crises and the process of industrial restructuring brought about by the exhaustion of the technological and growth systems of the 1950s and 60s. For its part, the expansion of *Computers* may be explained by the inception of information technologies based on the invention of the microchip at the beginning of the 1970s (Sundbo, 1998).

The case of *Computers* is interesting because it is doubtful whether this sector ends its cycle around 1986 as do the other sectors. As we have already pointed out, it does not present the saturation phase before this year and then presents an emergent phase. Is this only one single cycle till 2003? This may be interpreted in two ways:

- There was no structural break in the mid-1980s, and the period (1967-1986) may be viewed as a development phase for the cycle that appears in the second sub-period.
- There was indeed a structural break, because the first sub-period (1967-1986) shows an initial expansion that is interrupted by a change in technological conditions and demand in the sector.

However, we must recognize that the available information is not sufficient for us to discern which of these interpretations is nearer to the truth. It is likely that the explanation for this sectoral peculiarity contains something of each. In fact, both interpretations reflect well-known truths about the economics of innovation (see, e.g. Dosi (2000)). The 1970s laid the technological foundations for the development of activities related with information technologies and communications in the ensuing decades. One important aspect to note is that the technological development of ITT industries was not a smooth or uniform process, but rather was scattered with key innovations that had a profound impact on the technological and growth trajectory of the sector. Thus, invention of microprocessors opened the way to mass sales of PCs and, above all, to the internet revolution, events which would explain a possible

[5] It is not surprising that the econometric data obtained in Table 2 indicate that the fits in the first subperiod are not good (low R^2 except in *Computers* and non significant estimates). Only data on the final period is available.

structural break in the midst of the expansionary process that took place during the first period. According to this interpretation, the apparent break in the expansionary process would have paved the way for the accumulation of business infrastructure and new equipment, which extended throughout the 1990s, allowing for the introduction and spread of networking, cellular phones, global telecommunications, and so on.

Turning to the second sub-period, from February 1986 through 2003, the analysis reveals an S-shaped evolution in all sectors, with the rhythms and rates of growth taking the form of a bell biased toward the right. See Figures 9 and 10 and the values $b/d > 1$ in all cases, in Table 2. This indicates the existence of a period of expansion followed by contraction in both the total industry series and the sector series, with the emergent phase (and undoubtedly emergent phase + inflexion phase) lasting longer than the saturation phase. This can be verified in Table 2, supposing that January 2003 is the final date. These qualitative similarities should not, however, lead us to ignore the considerable differences that can be appreciated between some sectors and others in terms of the volume of investment and growth rates. The inter-sector differences are one of the most notable features of the expansion that took place in the 1990s (originating in the second half of the 1980s and perhaps, as in the case of *Computers*, resulting from a long process of sustained development reaching back into the 1970s). Investment and innovation in these years, according to the evolution of rhythms and rates of growth in Figures 8 and 9, was primarily oriented toward industrial transformation and the definitive consolidation of the sectors that emerged in the 1980s, rather than to mass production of known products (in contrast to the post-war system). The role of *Computers* as the engine of investment processes during this decade is unquestionable in light of our results. Moreover, Table 2 reveals fast growth in *Computers*, both via a wide floor-ceiling range in the second sub-period (over four times the range in the other series) and due to the high value of the a coefficient (over 8,400 times the value of a in the *USA CI for Total Industry* series). In other words, the high value of a is nothing other than a reflection of endogenous technological capacity in the sector, while its capacity to generate new activities and replace obsolete technologies would explain the significant gap between the floor and the ceiling.

The sectors which make up any economy are interdependent and industries have to grow at rates that are close to the average if they are not to disappear. Therefore, if a sector possesses a significantly higher endogenous growth capacity (a coefficient) than others, it is to be

If the series were completed with data from previous months, the estimates would improve and reveal the phases which are missing.

expected that growth related to demand and the business cycle will be stronger in the remaining industries. In other words, its ceiling and floor effects will be less significant. We should, therefore, expect to find an opposite relationship between the values of a and those of the floor and ceiling effect exponents b and d. This conjecture is confirmed by Table 2, where we find that the greater the value of a, the smaller the sum of $b+d$.[6]

As has already been pointed out, during the second sub-period, the ratio $b/d > 1$ in all cases confirms that the period of expansion is longer than that of recession; nevertheless the ratios differ depending on the sector. The highest is that of *Manufacturing*, followed by *Primary Processing*, then *Computers*, and *Durable Goods*. Consequently, we may affirm that the saturation effects (i.e. the impact of the ceiling) were more intense for *Durable Goods* and weaker in *Manufacturing*. In all cases, the saturation effects were significant toward the end of the sub-period because the dynamics were approaching their theoretical ceilings, the value of which reflects the exhaustion of growth. This exhaustion is in line with certain limiting forces observed by various researchers in relation to the growth regime of the US economy at the turn of the century (see, e.g. Gualerzi (2001)). The unequal distribution of incomes (which has intensified since the end of the 1970s), the polarization of the labor market (hindering job opportunities for the low-skilled), and the instability of the financial markets (the main investment option for American families) are just some of the disquieting factors that might explain an eventual fall in aggregate demand and the rhythm of industrial investment at the beginning of the 21st century.

5. CONCLUDING REMARKS

In this paper we have sought to overcome some of the limitations inherent in the standard logistic and Gompertz curves with respect the representation of certain economic phenomena, such as cycles and growth. The effectiveness of these curves in theoretical studies, which has motivated their application in some now classic contributions to our discipline, cannot hide their weaknesses when it comes to reproducing a number of observed facts in the empirical data, such as increasing and decreasing growth rates for certain macroeconomic variables, or the asymmetry of certain patterns of economic development.

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[6] It is impossible to find this result using conventional logistic or Gompertz functions because $b = d = 1$ in these functions.

The observation of these weaknesses has led us to look more closely at the formal properties of these curves and, in general, those of S-shaped dynamics, bearing in mind their application to both economics and other disciplines, for example demography. Specifically, we have defined a family of unimodal differential equations that not only include the conventional logistic and Gompertz curves, however they also cover practically the whole spectrum of sigmoid curves.

With this family it is possible to model any evolution in the form of S and distinguish the three phases which underlie these evolutions: emergent, inflexion and saturation, as well as the periods of expansion and contraction. Along the same lines, our generalization would allow us to give weight to the different phases of the evolution analyzed; specifically, the values of b and d show whether or not the effect of the emerging phase or the saturation phase is dominant.

We have selected three sub-families from within this general family that are both mathematically manageable and depend on five easily interpreted parameters, in such a way that any one of them may adequately formulate (from an empirical point of view) a wide range of S-shaped phenomena. In order to assess the power of these families to replicate real economic events, we have proceeded to calculate the fit with the *USA Capacity Index for Total Industry* (1967/2-2003/1) and the corresponding *USA Capacity Index* series for the *Durable Goods, Manufacturing, Computers* and *Primary Processing* sectors. The results for the remaining non-linear adjustments performed for the series are good approximations to the actual series in the second subperiod. As might have been expected from Proposition 3, the results for the three sub-families are fairly similar in terms of the quality of fit. In view of this, we have used the generalized logistic curve to extract information from the series, since it is the most easily interpreted of the three.

Based on the properties proved in Sections 2 and 3, and given the values for the parameters obtained in the adjustment calculations (Table 2), we have been able to draw various conclusions on the capital accumulation and investment patterns in the period analyzed that would appear to be in line with recent historical phenomena in the US economy. Thus, the period of decelerating growth in the 1970s, the recovery of the 1980s and the long expansion of the 1990s are all clearly distinguishable. Furthermore, the change in cycle around 1986 and the structural change which this involves are revealed in the major changes in the coefficients of the adjusted sigmoid dynamics. The evolutionary patterns differ widely between the sectors, with much more intensive growth visible in activities related with ITT. With regard to this sector, the fit also reveals a process of gestation during the 1970s and the possible existence of expansionary breaks in the development process resulting from fundamental innovations. Despite the diversity observed, especially in quantitative terms and (to some extent) at the turning points between the acceleration and deceleration of growth, there is a high degree of intersectoral similarity in the qualitative growth patters (bell-shaped in growth rates and rhythms and S-shaped in the original series). This highlights the existence of industrial interrelationships in the US economy. Finally, it may be of interest to note that our adjustments point to the exhaustion of growth in 2002, although the available information does not allow us to hazard a guess as to whether this will be temporary or permanent.

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APPENDIX

Proof of Proposition 1

Based on properties I-IV defining Ω, point 1 follows directly, as does the inference that the growth rate $\gamma(t) = \frac{\dot{x}(t)}{x(t)}$ is decreasing if $t > t^*$.

Let us consider two facts in order to appreciate the result if $t < t^*$. First, as $\dot{\gamma}(t) = \frac{g'(x)\,x - g(x)}{x^2}\dot{x}(t)$, we can be certain that $\dot{\gamma} >, =, < 0$ in $(0, t^*) \Leftrightarrow g'(x) >, =, < \frac{g(x)}{x}$ in (f, x^*). Secondly, if U is an open ball, then $g(x)$ is strictly convex (concave) if and only if $g(x) - g(y) > (<)\, g'(y)\,(x - y)$ for all $x, y \in$ U, $x \neq y$.

If $x_1 = f = 0$, we know from condition V of Ω that $g(x)$ is strictly concave in $(0, x^*)$. Therefore, for all $x(t) \in (0, x^*)$ we have that $g(0) - g(x) < g'(x)\,(0 - x)$ and, hence, $g'(x) < \frac{g(x)}{x}$ and $\gamma(t)$ is strictly decreasing in $(0, t^*)$.

If $x_1 = f > 0$, then $g'(x^*) = 0 < \frac{g(x^*)}{x^*}$, $g'(f) > 0 = \frac{g(f)}{f}$, and, hence, from Bolzano's theorem, a value $\hat{x} = x(\hat{t}) \in (f, x^*)$ therefore exists, such that $g'(\hat{x}) = \frac{g(\hat{x})}{\hat{x}}$. Furthermore, this $\hat{x}$ will be unique due to the strict concavity of $g(x)$. It follows that $\gamma(t)$ is strictly increasing in $(0, \hat{t})$ and strictly decreasing in $(\hat{t}, t^*)$.

If $x_1 > f \geq 0$, we know from condition V that $g(x)$ is strictly convex in (f, x_1) and, hence, for all $x(t) \in (f, x_1)$ we have that $g'(x) > \frac{g(x)}{x}$ and $\gamma(t)$ is strictly increasing in $(0, t_1)$, where $x(t_1) = x_1$.

We also know that $g'(x_1) > \frac{g(x_1)}{x_1}$ and $g'(x^*) = 0 < \frac{g(x^*)}{x^*}$ and, hence, from Bolzano's theorem and the convexity, that there is a single $\hat{x} = x(\hat{t}) \in (x_1, x^*)$ in which $g'(\hat{x}) = \frac{g(\hat{x})}{\hat{x}}$. Reasoning as before, it follows that $\gamma(t)$ is strictly increasing in $(0, \hat{t})$ and strictly decreasing in $(\hat{t}, t^*)$.

Proof of Proposition 2

Point 2 follows directly from 1 and from Proposition 1. In particular, point 2 of the latter determines the behavior of $\gamma(t)$ if $f = 0$ and $b \leq 1$ and point 3 of the remaining cases.

Point 1 is equivalent to the validation of the five properties of the Ω curves. The first three follow for the three sub-families. We shall, therefore, consider IV and V for each sub-family.

For the generalized logistic curves, because $g'(x) = a(x-f)^{b-1}(c-x)^{d-1}(b+d)\left(\frac{bc+df}{b+d} - x\right)$, we can be certain that $x^* = \frac{bc+df}{b+d}$ is the only root of $g'(x) = 0$ in (f, c). Also, because $g'(x) > 0$ if $x \in (f, x^*)$ and $g'(x) < 0$ if $x \in (x^*, c)$, IV is verified. Further, since

a) $g''(x) = a(x-f)^{b-2}(c-x)^{d-2}(b+d)m(x)$, with $m(x) = (b-1)(c-x)(x^*-x)-(d-1)(x-f)(x^*-x)-(x-f)(c-x)$,
b) $m(x^*) < 0$,
c) $m(f) > 0$ if $b > 1$, $m(f) < 0$ if $b < 1$ y $m(f) = 0$ if $b = 1$,
d) $m'(x) = 2(x-x^*)(b+d-1)$;

we can be certain that:

1) if $b > 1$, $g''(x)$ has a single root x_1 in (f, x^*), then $g(x)$ verifies V where $x_1 > f$.
2) if $b \leq 1$, $g''(x)$ will always be negative in (f, x^*) and V is verified where $f = x_1$.

Note that $x_1 > f \Leftrightarrow b > 1$ as affirmed in point 1.

In the cases of the generalized Gompertz and exponential curves, the proof of Properties IV and V is obtained by way of similar reasoning to that applied to the logistic curve, but starting, in the case of the Gompertz curve, from the basis that:

a) $g'(x) = a(x-f)^{b-1}(\ln((c-f)/(x-f)))^{d-1}(b\ln((c-f)/(x-f)) - d)$.
b) $g''(x) = a(x-f)^{b-2}(\ln((c-f)/(x-f)))^{d-2}m(x)$,
with $m(x) = [(b-1)\ln((c-f)/(x-f)) - (d-1)](b\ln((c-f)/(x-f)) - d) - b\ln((c-f)/(x-f)) =$
$b(b-1)(\ln((c-f)/(x-f)))^2 - (2b-1)d\ln((c-f)/(x-f)) + d(d-1)$

and, for the exponential curves, that:

a) $g'(x) = a(x-f)^{b-1}(e^c - e^x)^{d-1}z(x)$, with $z(x) = b(e^c - e^x) - d\,e^x(x-f)$.

b) $g''(x) = a(x-f)^{b-2}\,(e^c-e^x)^{d-2}\,m(x)$,

with $m(x) = b(b-1)(e^c-e^x)^2 - bd(x-f)(e^c-e^x)e^x - d(x-f)e^x[b(e^c-e^x)-de^x(x-f)] - d(x-f)^2e^{2x}$

$- d(x-f)^2(e^c-e^x)e^x = [b(e^c- e^x) - d\,e^x(x- f)]^2 - b(e^c-e^x)^2 - d\,(x-f)^2\,e^c e^x$

The value of x^* in point 3 has already been calculated. In order to obtain the values of $\hat{x}$ it is necessary to take into account that $\hat{x}$ must verify $xg'(x) = g(x)$, which again leads to $\hat{x}$ being the solution of $(b+d-1)\,x^2 - (bc+df-f-c)\,x - fc = 0$. It is simply a matter of calculation to obtain the various values of $\hat{x}$.

The value x^* in point 4 is obtained directly from the equation

$g'(x) = a\,(x-f)^{b-1}\,(\ln((c-f)/(x-f)))^{d-1}\,(b\ln((c-f)/(x-f)) - d) = 0.$

If, however, $b > 1$ and $f = 0$, since $\dot{g}(t) = \frac{g'(x)\,x - g(x)}{x^2}\dot{x}$, $\hat{x}$ is the solution to the equation $(b-1)\ln(c/x) - d = 0$, then $\hat{x} = c\,e^{-d/(b-1)}$.

Proof of Proposition 3

The conditions $f = f_d$ and $c = c_d$ are simply verified for any of the sub-families by choosing curves that have f_d as the floor and c_d as the ceiling. This can be done without any restriction.

Based on the equations that define x^*, it can also be proved for each of the three sub-families that there is always a curve that verifies $x^* = x_d^*$. To do this, it is sufficient to choose an appropriate value for $\mu = d/b$. In the case of the logistic and Gompertz curves, we may also make use of the values of x^* obtained in points 3 and 4 of proposition 2.

Furthermore, having established f, c, b and d, we may verify any of the three time conditions of the statement by choosing an appropriate value for a. If

$$\hat{T} = \int_{x_0}^{\hat{x}_d} \frac{1}{\dot{x}/a}\,dx,\quad T^* = \int_{x_0}^{x_d^*} \frac{1}{\dot{x}/a}\,dx \;\text{ y }\; T_F = \int_{x_0}^{x_F} \frac{1}{\dot{x}/a}\,dx,$$

it will be sufficient to choose $a = \hat{T}/\,T_d$, $T^*/\,T_d$, or $T_F\,/\,T_d$, for $\hat{t} = T_d$, $t^* = T_d$ or $T_F = T_d$, respectively.

Finally, let us consider how to choose b in each case to make $\hat{x} = \hat{x}_d$, assuming that $\mu = d/b$ is fixed, in order to verify the above properties. For each sub-family, if $\hat{x}_d = f_d$, we know from points 2 and 3 of Proposition 1 that $f_d = 0$. Therefore, it is sufficient, on the basis of point 2 of Proposition 2, to take $b = 1$ for $\hat{x} = \hat{x}_d = 0$.

In the case of the logistic curves, if $\hat{x}_d > f_d = 0$, for all $b > 1$ we have that $\hat{x} = \dfrac{c(b\text{-}1)}{b+d\text{-}1}$. Similarly, as $\lim\limits_{b\to 1} \hat{x} = \lim\limits_{b\to 1} \dfrac{c(b\text{-}1)}{b+\mu\, b\text{-}1} = 0 = f$ and $\lim\limits_{b\to +\infty} \hat{x} = \lim\limits_{b\to +\infty} \dfrac{c(b\text{-}1)}{b+\mu\, b\text{-}1} = \dfrac{c}{1+\mu} = x^*$ is verified and, hence, values for $b > 1$ exist, such that $\hat{x} = \hat{x}_d$.

Furthermore, if $\hat{x}_d > f_d > 0$, we know from point 3 of Proposition 2 that $\hat{x} = \dfrac{fc}{cd+fb}$ if $b+d = 1$, and $\hat{x} = \dfrac{bc+df\text{-}f\text{-}c+\sqrt{(bc+df\text{-}f\text{-}c)^2+4fc(b+d\text{-}1)}}{2(b+d\text{-}1)}$ if $b+d \neq 1$. Consequently, as

$$\lim_{b+d\to 1} \frac{bc+df\text{-}f\text{-}c+\sqrt{(bc+df\text{-}f\text{-}c)^2+4fc(b+d\text{-}1)}}{2(b+d\text{-}1)} = \frac{fc}{cd+fb},$$

$$\lim_{b\to 0} \frac{bc+df\text{-}f\text{-}c+\sqrt{(bc+df\text{-}f\text{-}c)^2+4fc(b+d\text{-}1)}}{2(b+d\text{-}1)} = f > 0,$$

$$\lim_{b\to +\infty} \frac{bc+df\text{-}f\text{-}c+\sqrt{(bc+df\text{-}f\text{-}c)^2+4fc(b+d\text{-}1)}}{2(b+d\text{-}1)} = x^* > 0,$$

we can be certain that values of b exist, such that $\hat{x} = \hat{x}_d$.

In the case of the generalized Gompertz curves, if $\hat{x}_d > f_d = 0$, then Proposition 2 tells us that for all $b > 1$ we have that $\hat{x} = c\mathrm{e}^{-b\mu/(b\text{-}1)}$. Since $\lim\limits_{b\to 1^+} \hat{x} = 0 = f$, and $\lim\limits_{b\to +\infty} \hat{x} = x^*$, we may select an appropriate value for b, such that $\hat{x} = \hat{x}_d$.

Similarly, if $\hat{x}_d > f_d > 0$, then equation $\ln((c\text{-}f)/(x\text{-}f)) - \mu = 0$ defines x^* and equation $bx[\ln((c\text{-}f)/(x\text{-}f)) - \mu] = (x\text{-}f)\ln((c\text{-}f)/(x\text{-}f))$ defines $\hat{x}$. Hence, we may assert that $\lim\limits_{b\to 0} \hat{x} = f$ and $\lim\limits_{b\to +\infty} \hat{x} = x^*$, and it is therefore possible to choose b, such that $\hat{x} = \hat{x}_d$.

In the case of the generalized exponential curves, we may begin with the fact that x^* is a root of equation $(\mathrm{e}^c\text{-}\mathrm{e}^x) - \mu\, \mathrm{e}^x(x\text{-}f) = 0$ and $\hat{x}$ of $bx[(\mathrm{e}^c\text{-}\mathrm{e}^x)\text{-}\mu\, \mathrm{e}^x(x\text{-}f)] = (x\text{-}f)(\mathrm{e}^c\text{-}\mathrm{e}^x)$. If $\hat{x}_d > f_d = 0$, then, since $\lim\limits_{b\to 1} \hat{x} = f = 0$ and $\lim\limits_{b\to +\infty} \hat{x} = x^*$, there exists a b that verifies $\hat{x} = \hat{x}_d$. Furthermore, if $\hat{x}_d > f_d > 0$, since $\lim\limits_{b\to 0} \hat{x} = f$ and $\lim\limits_{b\to +\infty} \hat{x} = x^*$, we may be certain that $\hat{x} = \hat{x}_d$ for the appropriate b.

Proof of Proposition 4

Point 1 follows directly from:

$$\frac{d\dot{x}(x)}{dx} = a[b(x\text{-}f)^{b\text{-}1}(c\text{-}x)^d - d(x\text{-}f)^b(c\text{-}x)^{d\text{-}1}] = a(x\text{-}f)^{b\text{-}1}(c\text{-}x)^{d\text{-}1}[b(c\text{-}x) - d(x\text{-}f)].$$

From point 2 we may infer the relationship:

$\dot{x}_2(x) = a(x\text{-}f)^{b_2}(c\text{-}x)^{d_2} = a(x\text{-}f)^{\lambda b_1}(c\text{-}x)^{\lambda d_1} = a\ [(x\text{-}f)^{b_1}(c\text{-}x)^{d_1}]^{\lambda}$.

The expression $\dot{x}_1(x^*)$ is obtained by direct calculation.