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Volatility, Heterogeneous Agents and Chaos

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Abstract

Agent heterogeneity, alongside several types of learning rules, has been used in recent economic literature to justify nonlinear dynamics for the time paths of aggregate economic variables. In this paper, the mechanism through which heterogeneous agents leads to chaotic motion is explained. Adding to a system with initial behavior heterogeneity an adaptive learning rule based on discrete choice theory, one is able to encounter a reasonable explanation for nonlinear motion. The adaptive learning / bounded rationality rule is not the only ingredient necessary for the absence of a long run steady state; heterogeneity must also imply that the several behavior possibilities alternate as the best behavioral choice. Only in such circumstances heterogeneity persists and an unpredictable outcome is likely to arise. After a review of the literature, the paper develops two models. The first is a generic approach that exemplifies how heterogeneity concerning the volatility of two stochastic processes may lead to chaotic motion; the second is a utility maximization setup, where the source of heterogeneity is investment decisions. For the utility problem, we find that the time path concerning consumption growth tends to stabilize around a constant value (a constant expected value is observable), but the steady state will be characterized by periods of low volatility that alternate with periods of high volatility.

Keywords: Heterogeneous agents, Bounded rationality, Chaos, Volatility.

JEL: C61, D91, E10

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1. Introduction

Economic agents exhibit distinct behavior patterns and do not share the same expectations about future events. This is a simple remark about reality, but it may have important implications in what concerns the evolution of aggregate economic variables over time. This paper focuses on the implications of behavior / expectations heterogeneity. General conditions under which heterogeneity implies significant changes in the evolution patterns of economic variables will be identified and a simple example of behavior heterogeneity describing a setup where consumers may choose between two investment opportunities will be developed.

The most significant issue involving agent heterogeneity relates to the fact that under certain conditions the interplay between the different types of agents could result in equilibria with unusual properties. This is specially true when considering the role of expectations in macroeconomics. The traditional analysis at this level uses the benchmark concept of rational expectations, but rational expectations leave no room for distinct behavioral rules. Thus, recent literature in macroeconomics is gradually adopting different views about expectations, where the two central concepts of the present analysis – belief heterogeneity and bounded rationality – are generally adopted to propose time series evolution explanations that are more likely to encounter compatibility with empirical evidence.

In the next sections, before presenting simple examples that support the idea of chaotic motion as a result of evolutionary competition, we identify the ways in which economists have been explaining economic nonlinearities in recent work. Generically, we might say that rational expectations are being replaced by a notion of adaptive learning. Adaptive learning may be interpreted in several ways and can be thought of in a scenario of homogeneous behavior or, alternatively, in an environment populated by agents with different strategies and beliefs. This last view is chosen for the subsequent analysis because complicated behavior and nonlinear dynamics are easily associated to the notion of absence of homogeneity.

For instance, as one will see in the following sections, in asset pricing theory the concept of heterogeneity arises as a central piece in the explanation of economic behavior. Because present prices in financial markets are dependent upon expectations about future prices, the co-existence of different expectation rules can lead to price movements that are erratic and impossible to predict, since in different time periods some type of expectation may dominate the market, while in other moments other expectation rules will determine price evolution. In the following sections, asset pricing will not be a central concern of our analysis, but the way in which agent heterogeneity is approached in this literature will be a fundamental reference to our arguments. In particular, the path breaking work by Brock and Hommes (1997, 1998), where heterogeneous expectations are linked to an adaptive belief system of bounded rationality serves as an important guiding line. These two authors are the founders of the 'rational routes to randomness' concept, a concept that explains the way in which individuals that are rational but have different beliefs about future events may imply nonlinear evolution of economic aggregates (namely, in what relates their main concern, asset prices).

In the following sections we will emphasize that the absence of a homogeneous behavior among agents with the same economic goals is an important route leading to chaos; nevertheless, it is not the only one – models in various fields of economic analysis produce strange dynamics without taking heterogeneity as a nuclear starting point. We address this issue in section 2.

The remainder of this paper is organized as follows. Section 2 presents an overview of the literature relating bounded rationality, learning and behavioral heterogeneity. Section 3 formalizes the idea that behavior heterogeneity is not the only ingredient necessary for an adaptive system leading to nonlinear dynamics; a second feature is equally important – the absence of a fully rational scenario. The concept of bounded rationality in a discrete choice model is developed. Section 4 illustrates with a simple example how agent heterogeneity can lead to unpredictable economic behavior over time. Section 5 elaborates a more sophisticated example with important economic meaning. An intertemporal utility maximization setup is considered and agents will differ in the portfolio investments they undertake; agents will have distinct consumption opportunities, given that they will obtain different returns from their wealth. The main result about this model is that the existence of distinct investment opportunities implies, under the model's features, an aggregate long run consumption path that is not smooth and predictable. Finally, section 6 systematizes the most relevant conclusions.

2. Expectations and Adaptive Learning

The representative agent framework is the benchmark tool for macroeconomic analysis. Such a setup presents several desirable features. First, it allows for an intuitive and simple treatment and discussion of the dynamic behavior of most economic aggregates, and, in this way, it presents itself as an effective instrument for the analysis of economic policy and welfare implications of macroeconomic shocks. Second, the representative agent framework implicitly considers fully rational behavior and expectations. The agents make decisions, under some resource constraints, in order to optimize a given objective function that determines future outcomes (the optimization behavior suggests choosing rationally). Furthermore, to evaluate future results it is necessary to form expectations about predictable outcomes; in this respect, the representative agent paradigm relies on Muth (1961) in assuming that all predictions about future events make use of all the available information in the present state, which implies that the agent expectations will coincide with the prediction of the relevant economic theory.

As Kirman (1992) stresses, although appealing, the representative agent framework has a rather limited ability to explain real aggregate economic phenomena. Therefore, economic theory should evolve in the direction of building more convincing and complete explanations of human dynamic behavior, in order to get to more insightful and realistic results about aggregate dynamics. This is indeed the path followed in recent macroeconomic research. The concerns include both issues that were identified above:

(*i*) economists are searching for alternative ways of presenting agents' formation of expectations; rational expectations do not clearly represent a realistic thinking mechanism, in the sense that these are too demanding, they imply strong cognitive capabilities, powerful computational skills and the absence of any relevant errors in the decision making process;

(*ii*) although tractable and appealing, the idea of a single agent as a representation of the multiple possible kinds of behavior that agents may adopt in the economic system is an exaggerated and unrealistic assumption. The economic activity is shaped by diversity of behavior, expectations and preferences; taking a representative agent would mean that all individuals tend to be similar in their choices. This is not true; individuals respond to different stimuli and they have distinct experiences with respect to past results regarding their choices, and, as a consequence, they will not all behave or form beliefs in the same way.

It is obvious that there is a close link between the absence of absolute rational behavior and beliefs and agent heterogeneity. Under a heterogeneous agents scenario, full rationality implies that individuals would be able not only to make their own informed predictions about coming events, but also that they could form unbiased predictions about the behavior of all the other agents. In such circumstances, heterogeneity would vanish because all agents would adopt, for a given scenario, the only admissible behavior rule: the rational one. This is in fact the argument of the supporters of the rational representative agent paradigm: there is no need to consider agent heterogeneity because rationality is linked to the best market outcome and thus all behavior and beliefs will converge to the single optimal behavior / belief. Friedman (1953) pointed out that markets are mechanisms of behavior adjustment that tend to reward rational agents and penalize 'irrational' behavior. The agents that do not comply with the market performance rules are driven out of the market as a result of the poor performance; the agents that accept the signs given by the market will learn to behave rationally and heterogeneity dissipates.

In accordance with the previous argument, the representative agent paradigm relies on rationality to exclude heterogeneity. Recent research adopts the opposite view: the evidence about heterogeneous behavior and beliefs is an important clue pointing to the absence of such a thing as a unique notion of rational behavior and rational prediction capabilities.

If one wants to understand the behavior through time of macroeconomic aggregates it is necessary to search for the true mechanism through which individuals form their expectations. It became clear, with the previous discussion, that individuals are not born with a complete knowledge of their environment and in this way they are not able to make a fully informed choice in each time period. Hence, agents have to learn and adapt through time according to the evolution of the economic system, and so they will adjust their behavior in each time moment updating their beliefs as new information arrives to their knowledge.

The main difference between the rational expectations approach and the new 'bounded rationality' paradigm, is that under this new setup the agents beliefs feed back to the behavior of economic variables. Full rationality implies that individuals know how the economy truly works and, therefore, they will adapt the correspondent behavior to the motion of economic aggregates, contributing for such behavior to be perpetuated. Differently, under an adaptive perspective, agents look to the present state of the economy and attach the new information to the beliefs they already had, what may allow to change their own behavior and also the aggregate behavior. In this way, aggregate variables will not display, eventually, a trivial and smooth time path; such a time path is determined in each moment by the competition of conflicting strategies that the agents exhibit.

The unpredictable nature of the time series of many aggregate economic variables has a candidate justification on the existence of an evolutionary competition environment where agents with heterogeneous beliefs interact. Agents have distinct initial beliefs and so they will add new information in different ways to their knowledge, which implies different behavior strategies. This 'artificial intelligence' process leads to an infinite number of alternatives regarding behavior changes, which certainly implies unpredictable results regarding the movement through time of economic time series.

An important motivation underlying the bounded rationality – heterogeneous agents is precisely the need for encountering reasonable explanations for the erratic movement of many macroeconomic aggregates. Although nonlinear dynamics and chaotic behavior may be supported under a full rational representative agent setup, as in Grandmont (1985), regarding business cycles analysis, and Benhabib, Schmitt-Grohé and Uribe (2001*a*, 2001*b*, 2001*c*) regarding monetary policy and the interest rates Taylor rules, the truth is that the bounded rationality – heterogeneous agents setup is an intuitive environment for the discussion of non trivial dynamic motion.

Note that, despite the effort of the new literature on macroeconomic dynamics in replacing rational expectations by a more realistic beliefs formations setting, the notion of rationality remains strong. Individuals continue to make use of every piece of information they possess to change their behavior in the direction of a better performance result. In this way, macroeconomic expectations literature does not intend to use the concept of bounded rationality in exactly the same sense as Simon (1955, 1959, 1982) has developed it. Recent research has pointed to the bounds and limits of human cognition. For instance, Romer (2000) highlights the importance of emotions in decision making; feelings like desire, envy or greed are certainly present in every individual decision but are difficult to capture and to include in an explanation of aggregate behavior. Also, Gabaix and Laibson (2004) propose a thorough analysis of the decision making process, giving special attention to the effort that is associated with weighting costs and benefits of choosing some particular option. It is as well relevant to be aware that human cognition is not perfect; Kahneman (2003) refers to this problem, explaining that intuition tends most of the time to be preferred to an exhaustive investigation about available choices. The problem is that intuition makes use of past results to evaluate present decision problems that many times need a stronger cognitive effort than the simple adaptation of known rules.

In the sense that cognitive deficiencies, emotions and other peculiarities of human decision making are not considered in detail, the analysis in macroeconomic bounded rational – heterogeneous agents contexts continues to be an analysis based on rationality, but where rationality involves managing incoming information and not the adoption of a simplistic view that information is instantly available to make decisions.

A more detailed analysis implies identifying alternative evolutionary competition approaches. One of the most popular approaches at this level relates to learning models. In learning models, as exhaustively studied by Sargent (1993), Evans and Honkapohja (2001) and Tuinstra and Wagener (2003), among others, the presence of heterogeneity is not essential to address the bounded rationality concerns. The key point in learning models is the presence of an adaptive learning mechanism, under which it is assumed that economic agents act as staticians or econometricians when they make predictions about future economic events. Adaptive learning means that agents use new available data to adjust or update, at each moment in time, their forecast rules.

The formation of expectations through adaptive learning is one of the most influential fields of research in macroeconomics today. Important recent developments in this area include Evans and Honkapohja (2003*a*, 2003*b*), Honkapohja and Mitra (2003*a*), McCallum (2002) and McGough (2003). These references contribute to the theory of adaptive learning, finding support for the possibility of nonlinear dynamic behavior regarding economic aggregates. The learning approach considers the possibility of convergence to the rational expectations equilibrium as a special case, but this is far from being the rule. Irregular, unpredictable or chaotic motion become strong possibilities as a result of the boundedly rational setup, in which agents form expectations based upon observation and adapt the forecasting results with the incoming additional information.

Learning as a source of nonlinear dynamics is being used on applied studies in several areas of economic analysis. Namely, economic policy, which makes extensive use of the expectations concept, is gradually replacing the notion of rational expectations by some kind of learning rule to present more credible explanations of the impact of policy decisions over the aggregate behavior of economic variables. The most widely studied field at this level is monetary policy.

The rebirth of monetary policy as a powerful mean to explain short-run fluctuations, that has strongly benefited from the influential work on the Phillips curve relation under sticky prices by Clarida, Gali and Gertler (1999) and Woodford (2003), has led to an unavoidable interest in the perception of the ways in which the formation of expectations influences short term economic activity. The new Keynesian Phillips curve implies the necessity of considering expectations about future inflation and the need for understanding that rational expectations have a limited ability to link theoretical results with empirical observation. It is by including a process of adaptive learning under bounded rationality that the new approach to monetary policy turns into a powerful tool for understanding economic fluctuations. Among the extensive literature on learning and monetary policy one may highlight the work of Bullard and Mitra (2002, 2003), Evans and Honkapohja (2003*c*), Honkapohja and Mitra (2003*b*, 2005) and Preston (2004).

Besides monetary policy, the role of expectations has been also considered in other areas of economic policy. In particular, fiscal policy under learning is the subject of study in Evans and Honkapohja (2002, 2003*d*), while foreign exchange policy is addressed by Cho and Kasa (2002) and Kim (2003).

Learning implies adaptation through time and may transform a stable outcome (under rational expectations) into a nonlinear system leading to unpredictable time series evolution. This happens without the need to assume that agents behave and form expectations in different ways – individuals make their predictions as econometricians as new information arrives, but they can act in a same way. Nevertheless, the consideration of heterogeneous agents under learning introduces a competition component that allows for a closer link between the analytical setup and the empirical evidence. In fact, some of the most influential adaptive learning studies make use of a setup where agents have different characteristics in the way they form expectations. This is the case of Branch and McGough (2004), Giannitsarou (2003), Guse (2003) and Honkapohja and Mitra (2002). The introduction of heterogeneity in beliefs and expectations formation implies a new degree of complexity and it makes more likely to find a dynamic behavior for economic variables that depart from the stability concept.

The term adaptive learning allows to distinguish from other types of learning processes that are considered in the literature. Briefly, some of the most relevant are now identified:

i) Eductive learning. The notion of eductive learning relates to the idea that a mental process of reasoning leads to the coordination of expectations among agents, allowing to achieve a rational expectations equilibrium. This means that eductive learning is somewhere in the middle between a rational expectations assumption, in which individuals perceive instantly which is the best choice relating some future outcome, and adaptive learning, that is associated with a never ending process of behavior updating. Eductive learning is based on the notion that agents coordinate expectations and that they are able to find a shortcut through which expectations converge to a stable equilibrium. Evans and Guesnerie (2000, 2003) discuss the eductive learning approach.

ii) Genetic algorithms. Models using genetic algorithms relate to computational intelligence. In the context proposed by these models, a specific artificial device is used to memorize and reproduce patterns of behavior. Specifically, the concepts of reproduction, crossover, mutation and election are used. These four genetic operations mean, under a learning perspective, that agents learn by: imitating well succeeded actions (reproduction); testing new alternatives (crossover); looking at different alternatives (mutation); and selecting the potentially more favourable possibilities (election). Genetic algorithms have been brought to the domain of learning devices in economics for instance by Arifovic (1994), Bullard and Duffy (1998, 1999) and Casari (2003).

iii) Neural networks. This approach relies as well on computational intelligence. It makes use of a network where several nodes (the neurons) are connected. Through such connections, information is exchanged and efficient outcomes are retained. In economics, Beltratti, Margarita and Terna (1996) and Cho and Sargent (1996) are important references in applying these networks to create processes of learning under bounded rationality.

iv) Rational beliefs. Another approach to macroeconomic learning involves the concept of rational beliefs equilibrium, developed in Kurz (1994, 1997) and Kurz and Motolese (2001). According to Kurz, Jin and Motolese (2003), page 191, "the RB theory *assumes* that agents do not know the true probability underlying the equilibrium process but that they have a great deal of past data about the observable variables in the economy. Using past data agents compute the empirical distribution and construct from it a probability measure over infinite sequences of observable variables." The rational beliefs approach departs from adaptive learning in the sense that in rational beliefs there is not an important concern about how new information changes beliefs, because this new information has a small weight on the 'great deal of past data' individuals possess.

v) Discrete choice theory. It is on discrete choice theory that we will rely on to discuss, in the following sections, nonlinearities and chaos. The mechanics proposed by discrete choice models puts together the two essential features that are in the core of our discussion: bounded rationality, and thus a learning rule (which is also the central piece of the previously referred approaches to the formation of beliefs) and agent heterogeneity. Discrete choice is associated to the notion of competing strategies and to the idea that agents choose to change strategy according to a multinomial logit probability.

3. Heterogeneity and Evolutionary Competition

Initially developed by McFadden (1973), Manski and McFadden (1981) and Anderson, de Palma and Thisse (1993), the discrete choice setup has been brought to economic and financial analysis by the hands of Brock and Hommes (1997, 1998). The 'rational routes to randomness' literature that these authors have initiated rejects Friedman's concept of dominance of rational behavior, replacing the representative agent framework by a world populated by behaviorally heterogeneous agents.[1]

This approach acquired the designation of 'adaptive beliefs'. On an adaptive belief system (ABS), economic agents are, in an initial state, heterogeneous, i. e., they have different beliefs about how the economic system will evolve. The heterogeneity does not tend, however, most of the times, to vanish over consecutive periods, although individuals are allowed to change their beliefs as accumulated results imply that is rational to change from the selected strategy to another strategy that reveals a higher degree of performance. In such a system, the long run scenario is not likely to be an equilibrium predictable path. Rather, it will tend to characterize a complex evolving system, where the nonlinear (eventually chaotic) long term behavior of economic aggregates is the result of the interaction of many different

[1] Note that in this strand of economic thought, linked to the perception of macroeconomic behavior, we are referring to behavioral or belief heterogeneity. Other kinds of heterogeneity in agents' characteristics can be considered. For instance, Azariadis and Kaas (2002) study a standard intertemporal consumption utility maximization model in which agents distinguish from one another in what concerns time preference, that is, they have distinct discount rates relating future consumption.

boundedly rational agents, that possess distinct strategies and expectations that co-evolve over time implying a result that is most of the times an unpredictable result.

The discrete choice – ABS framework has its most relevant and popular application on the analysis of financial markets. The following is a small list of important references relating the use of the ABS setup in trying to understand the financial markets functioning and in particular the movement of stock prices through time: Brock and Hommes (2001), Brock, Hommes and Wagener (2001), Chiarella and He (2002), Gaunersdorfer, Hommes and Wagener (2003), Hommes (2001, 2005) and Hommes, Sonnemans, Tuinstra and van de Velden (2005).

The common feature to these references consists in attributing the explanation of asset price movements to the identification of two types of asset traders. The distinction is made between fundamentalist traders – who base their forecasts of future prices and returns upon economic fundamentals, such as dividends, interest rates or growth indicators – and chartists or technical analysts – these look at patterns of past price behavior and extrapolate trends from such patterns. The orthodox economic theory would suggest that only the first type of agents act rationally, and since the asset prices tend to return to a fundamental value, technical traders would end up by realizing that they were not following the most profitable strategy, and therefore in the long run only fundamentalists would subsist in the market, validating in this way the efficient market hypothesis of Fama (1970).

The discrete choice approach does not confirm that one strategy tends to perform better than the other systematically over time. The evolutionary competition among trader types and the chosen strategies tends to lead to an irregular switching between the different strategies, what culminates in erratic, complicated, even chaotic asset price movements. The absence of a tendency for one of the trader types to concentrate all the traders will mean that the market is characterized by a "perpetual evolutionary switching between competing trading strategies" [Hommes (2004), page 5]. In this way, the notion of rationality becomes blurred – who is in fact rational: a fundamentalist trader that tends to perform poorly, or a technical analyst that withdraws the most substantial gains? This type of analysis rises the doubt about rational behavior and suggests that it is not the economic model that determines rationality but rather it is concrete evolution that really determines who is right.

The evolutionary dynamics underlying the ABS model has gained influence as an explanation of stock price movements because it does not only provide an intuitive explanation for the behavior of traders and for the irregular pattern of price evolution, but also because the model is capable of mimic some of the most relevant stylized facts often observed in financial series. In particular, such a procedure of analysis is capable of simulating series with volatility clustering, unpredictable returns and long memory, which correspond to actual characteristics of observed price movements in financial markets. Deviations from the fundamental price may be triggered by uncertainty about economic fundamentals, but such deviations are amplified by the evolutionary interaction among competing trading strategies followed by boundedly rational agents: this is what the model intends to capture and it is as well what real markets apparently involve. In this way, the ABS structure is able of taking real world regularities of market behavior to explain real world evidence about economic time series.

Other applications of the discrete choice framework have been developed. Namely, in what concerns exchange rate time series, a same kind of separation between fundamentalists and speculators is considered by Manzan and Westerhoff (2005) and De Grauwe and Grimaldi (2005). Their results are similar to the ones in the ABS financial markets literature; specifically, evolutionary competition among traders in the foreign exchange market leads to an irregular and unpredictable motion of the exchange rate through time.

Through the present section we have placed alongside two important features of human behavior. Economic agents are boundedly rational and they tend to choose different behavioral approaches when they predict future events. Bounded rationality can be interpreted in multiple senses; it may mean that agents are not capable of processing large quantities of information and thus they tend to choose relatively simple strategies in their actions; it can also mean that individuals, even if they have strong cognitive skills, do not have access instantly to all the information about a given environment, and therefore they will have to update their behavior as new information arrives. We have argued that bounded rationality may constitute a route to chaos or a route to randomness but it may also imply a process of convergence to a rational expectations long-run locus.

The route to randomness is reinforced by the assumption of a tendency of the economic system to allow for coexistence over time of different types of behavior and beliefs. As Diks and van der Weide (2003) argue, heterogeneity is a natural source of randomness, in the sense that the unpredictability of economic time series tend to rise as the number of alternative trading strategies is enlarged and as the dependence among agents' choices becomes more complex. Furthermore, Barucci (1999) and Negroni (2003) realize that the introduction of heterogeneity leads to high dimensional systems where a chaotic pattern of evolution for economic variables might arise.

Then, our main point is the following: heterogeneity of beliefs, bounded rationality, adaptive learning and evolutionary competition are best understood if taken together. Connecting these terms implies, most likely, to find natural nonlinearities (i. e., nonlinearities that are endogenous and not driven by external shocks). The discrete choice model seems to capture the fundamental four characteristics pointed above. For this reason, the next sections make use of this analytical structure to illustrate the way in which endogenous nonlinearities might arise. The examples are extremely simple: they just consider differences in time series volatility.

According to last paragraphs' analysis, heterogeneity simply means that individuals or some kind of groups will not all behave in the same way. As discussed, to conventional economic theory such heterogeneous behavior does not tend to persist. After all, if two groups of agents act distinctly concerning some economic phenomenon, one of these groups will have better results and all individuals will change to the better performance group. Rationality means in this way the absence of persistence of all actions besides the one that gives the best result.

Thus, heterogeneity is unlikely to hold under a fully rational setup, because agents certainly do not hold on to a behavior or belief that performs poorly. The key point in favour of the idea of heterogeneity persistence is that individuals are not seen as completely rational, instead they follow some kind of bounded rationality rule, that introduces some sluggishness in the way each individual changes his behavior.

The concept of bounded rationality to adopt is linked to the discrete choice theory literature. Discrete choice relies on a mechanism through which agents change behavior over time, without eliminating heterogeneity; the shares of individuals attached to each behavioral group are updated in each time moment and the evolution of such shares implies an everlasting change in the correspondent values.

We define n_{ht} as the share of individuals that follow some kind of behavioral rule h in a moment of time t. In the presence of H possible rules, we will have H shares. We are interested in the way each percentage n_{ht} evolves over time. Discrete choice theory describes a fitness function or performance measure U_{ht} relating to each of the possibilities of behavior.

Individuals will change from one alternative behavior to another according to the value of U_{ht}. The better the results given by the chosen strategy of action / behavioral rule, the faster agents will change from anyone group to the best performance group. Discrete choice points to the following expression as representing the percentage of individuals attached to behavior group h in a moment of time t+1:

$$n_{h,t+1} = \frac{e^{\beta . U_{ht}}}{\sum_{h=1}^{H} e^{\beta . U_{ht}}} \tag{1}$$

In expression (1), it is clear that the better the performance of the strategy h relatively to all the other strategies (measured by the U_{ht} functions), the higher will be the value of the share. Simultaneously, this share depends on a parameter $\beta \geq 0$. The parameter β is the intensity of choice and it is a measure of the sensitivity of the agents to the differences in results of the various rules of behavior. A high β means that individuals change behavior rapidly as other U functions display better results than the U function attached to the behavior followed in a given moment. A value of parameter β close to zero implies that individuals have more resistance to change and stick with the same strategy of action, even if this does not perform as good as other strategies for several consecutive periods of time. In this way, parameter β is a measure of the time needed for individuals to realize that it is not worth to keep with a behavior that tends to produce worse results than other behaviors.

In other words, β is a measure of bounded rationality. An increase in the choice parameter value represents a faster response to better incentives (note that if β=0, the agents will have an extreme behavior of accepting no change and thus $n_{h,t+1}=1/H$, $\forall$ h=1, …, H). A high β may represent a high level of rationality, in the sense that agents are fast in changing their behavior to a best performance option. Nevertheless, a high value of the intensity of choice can also involve rationality deficiencies, because choices will not take into account the fundamentals but simply an evaluation of past results. In particular, for short memory environments only recent realizations are taken into account, and thus expectations are purely adaptive or even naive.

Combining this interpretation of bounded rationality with the existence of several alternatives concerning behavior or beliefs, we will be able to encounter a setup under which one finds an unpredictable time path for the variables underlying such behavior choices. The interesting point is that two or more perfectly understandable time evolution mechanisms, when combined with bounded rationality may result on a time path that is erratic, impossible to predict and has traces of chaos.[2]

Looking at expression (1) it is straightforward to understand that $n_{h,t+1}$ converges to 0 or 1 if the performance of strategy h is worse or better, respectively, than any other possible strategy. Thus, heterogeneity will not hold in the long run under the scenario of a systematic difference in performance of the possible alternative actions.

In this way, we have found two conditions that are essential for agent heterogeneity to persist and thus to have eventually a meaningful impact on aggregate economic behavior:

(1) it is necessary to consider a bounded rationality approach, in the present case based on discrete choice theory. Under such a setup agents will change behavior but not instantly. They will look to accumulated results of their strategic choice and they will change it when such accumulated results manifestly point to the other strategy as leading to better results;

(2) given two or more strategies or behavioral rules, the outcome of one cannot be systematically more favourable than the other(s). Functions U_{ht} of accumulated results should intercept systematically over time to avoid the predominance of one of the strategies, case in which heterogeneity will end up disappearing.

4. A Basic Setup

In this section we present a basic example of persistence of agents heterogeneity. We take an undetermined number of agents that may choose between two strategies. The first gives, in each time moment, an unknown result with an expected value of μ and a variance σ_i^2. The second result is also a stochastic process with a mean of μ and a variance $\sigma_j^2 \neq \sigma_i^2$. That is,

[2] We understand chaos as the situation under which ‘a pair of initial values located arbitrarily close together may lead to completely different time series though they are generated by the same dynamical system.’ [Lorenz (1997), page 119].

$$\delta_{it} \sim iid(\mu, \sigma_i^2) \tag{2}$$

$$\delta_{jt} \sim iid(\mu, \sigma_j^2) \tag{3}$$

The time paths of (2) and (3) are easy to describe. We do not know in each moment of time which value we will have, but over a time interval with some observations we verify that both time paths are constituted by points around μ that may be further away from this value if the variance parameter has a higher value. These time series, concerning the two choices each agent faces, respect the second of the conditions presented in the last section for heterogeneity to be meaningful. Certainly best results will alternate: in some moments of time (2) will exhibit a higher value than (3), but in other moments the opposite is true.

We assume that the goal of the individuals in the economy is to choose the time series in (2) or the time series in (3) that allows to obtain higher values. They will not change the choice they make between one of the two possibilities instantly, but they will evaluate results according to a fitness function. Let the fitness function in this case be the sum of all the past results and the present result, where past results are discounted at a rate ρ_δ (individuals value recent results more than results far away in the past),

$$U_{it} = \sum_{s=0}^{t} \delta_{is} \cdot \frac{1}{(1+\rho_\delta)^{t-s}} \tag{4}$$

$$U_{jt} = \sum_{s=0}^{t} \delta_{js} \cdot \frac{1}{(1+\rho_\delta)^{t-s}} \tag{5}$$

with t the present moment.

The discrete choice model will in this case state that the following expression gives the number of individuals that stick with results (2) in each time moment (recall that this will change faster or slower according to the value of the intensity of choice parameter, β),

$$n_{i,t+1} = \frac{e^{\beta . U_{it}}}{e^{\beta . U_{it}} + e^{\beta . U_{jt}}} \qquad (6)$$

Our concern is with the average value of (2) and (3). If n_{it} were a constant value, such a time series would present a behavior similar to (2) and (3), with a volatility that would be somewhere between σ_i^2 and σ_j^2. But n_{it} changes at every time moment, and thus $\zeta_t = n_{it}.\delta_{it}+(1-n_{it}).\delta_{jt}$ will exhibit an erratic, completely unpredictable (or chaotic) behavior. Our conclusion is that in the presence of two results, giving the possibility of each agent to choose the strategy that best performs under a discrete choice framework, the overall result (the weighted average of the two results) ends up by being a time series with clusters of different volatility and thus completely unpredictable.

This is a general setup, and (2) and (3) may be anything. Section 5 concretizes these series as being the returns from financial assets. In appendix [I] the results of the setup of this section are presented graphically for concrete values of parameters and for a normal distribution; the parameter values are μ=0, β=2 and ρ_δ=0.01, with $\sigma_i^2<\sigma_j^2$ (in this case, let σ_i=0.1 and σ_j=1). The graphics display three of the infinite time paths that can be presented for δ_{it}, δ_{jt}, n_{it} and ζ_t. As one can observe, the behavior of δ_{it} and δ_{jt} follow a same pattern in each case, but in reality they give rise to time paths n_{it} and ζ_t that are enormously different for each time we run the example. For ζ_t, periods of high volatility co-exist with periods of low volatility, which reflects the predominance of one of the two series [(2) and (3)], nevertheless it is unpredictable which of the series will dominate in each moment and how the change of predominance is realized.

5. Utility Maximization and Heterogeneous Portfolios

Consider now a standard intertemporal optimization model regarding consumption utility. Instead of a representative agent assume two types of agents in what concerns investment decisions. A first type of agents includes the ones that invest their wealth in a risk free asset. The second group invests in a risky asset with an expected income rate equal to the income of the risk free asset. Considering a utility function that exhibits decreasing marginal utility and a discount factor $\rho>0$, agents are distributed between the two types and the following problems are assumed:

- Agents of type *i*:

$$Max\sum_{t=0}^{\infty}U(c_{it}).\frac{1}{(1+\rho)^t} \text{ subject to } a_{i\,t+1}=(1+r).a_{i\,t}-c_{i\,t},\ a_{i0} \text{ given;}$$

- Agents of type *j*:

$$Max\sum_{t=0}^{\infty}U(c_{jt}).\frac{1}{(1+\rho)^t} \text{ subject to } a_{j\,t+1}=(1+r+\delta_t).a_{j\,t}-c_{j\,t},\ a_{j0} \text{ given.}$$

In these problems, c_{ht}, $h=i, j$, defines the consumption level of each type of agent, a_{ht}, $h=i, j$, is the wealth of an individual in group h, and r is the rate of return of the wealth not consumed in each period (expected rate of return in the case of individuals in the *j* group). We ignore any fixed return (not dependent on the wealth endowment). We consider that the rate of return on wealth of individuals of type *j* has a stochastic component $\delta_t \sim iid(0,\sigma^2)$.[3]

Solving the optimal control problem for both agent types, we reach a constant consumption growth rate for agents of type *i*, and a consumption growth rate with a same constant expected value for agents of type *j* but where a volatility component is present,

[3] Note that, in the present analytical setup, optimal control problems relating to agents of type *i* and agents of type *j* do not correspond to the actual intertemporal problem faced by each single agent. They respect to the problems agents would face if they never changed behavior. Therefore, the pair of presented problems will give rise to two consumption growth rates and the problem of each agent consists in evaluating the outcomes referring to each solution, choosing the one that gives higher accumulated outcomes.

$$\frac{\Delta c_{it}}{c_{it}} = \frac{1}{\theta}.(r-\rho) \tag{7}$$

$$\frac{\Delta c_{jt}}{c_{jt}} = \frac{1}{\theta}.(r+\delta-\rho) \tag{8}$$

with θ>1 a concavity parameter of the utility function.[4]

The growth rate of the consumption aggregate is a weighted average of the growth rates (7) and (8). The shares of individuals choosing one of the two investment strategies are determined by a rule like (6), with U_{it} and U_{jt} the fitness functions, defined in terms of utility results,

$$U_{it} = \sum_{s=0}^{t} U(c_{is}).\frac{1}{(1+\tau)^{t-s}} \tag{9}$$

$$U_{jt} = \sum_{s=0}^{t} U(c_{js}).\frac{1}{(1+\tau)^{t-s}} \tag{10}$$

In (9) and (10), τ represents a memory parameter (past utility is important for the evaluation of the best investment strategy, but the farther away in the past is the utility result, less valued it will be today). The growth rate of aggregate consumption is then

[4] The derivation of (7) and (8) is made computing optimality conditions for the problems of agents of type *i* and *j*, respectively. For instance, to accomplish growth rate (7), one considers the current value Hamiltonian function $H(t) \equiv U(c_{it}) + p_{it}.(r.a_{it} - c_{it})$, with p_{it} a co-state variable. Optimality conditions are: (i) $H_c = 0 \Rightarrow U' = p_{it}$ or $c_{it}^{-\theta} = p_{it}$ when considering the concavity parameter θ; (ii) $\Delta p_{it} = (\rho - r).p_{it}$; (iii) $\lim_{t\to\infty} p_{it}.\frac{1}{(1+\rho)^t}.a_{it} = 0$ (transversality condition). Combining (i) and (ii), it is straightforward to derive (7).

$$\frac{\Delta c_t}{c_t} = \left(\frac{\Delta c_{it}}{c_{it}}\right)^{n_{it}} \cdot \left(\frac{\Delta c_{jt}}{c_{jt}}\right)^{1-n_{it}} \tag{11}$$

For (11) we can expect the same kind of lack of constancy (or alternate high and low volatility) as for the ζ_t series in the previous section. The conclusion is that in an economy where agents may choose between applying their savings in risky assets or, alternatively, risk free assets, and there is bounded rationality, the aggregate consumption growth rate will display an erratic / chaotic behavior that is impossible to predict in the initial moment.

To illustrate the previous logic we consider a numerical example. Assuming the following parameter values, $\{r, \rho, \theta, \tau, \beta, \sigma\}$={0.05; 0.01; 2; 0.01; 20; 0.01}, and a normal distribution for δ_t, appendix [II] presents three of the infinite results that are possible for (11).

6. Final Remarks

Agents look at reality from different perspectives. For example, they are risk averse in their investment decisions in different degrees. The economic science had always the conscience about this fact, nevertheless it was never seen as a fundamental source of disturbance over the time path of important economic variables. The mainstream economic thought is based on a notion of rationality that does not leave space for anything more than the choice for the best result attainable at any moment of time. The main rule is that agents, based on available information, have the ability at any moment to choose the behavior that produces the best expected result.

Bounded rationality and discrete choice behavior intends to add an element of inertia to decisions, which is in reality present in many of the economic decisions individuals make. Now, it is under consideration not only the best instantaneous result, but also the way the agents psychologically weight how their behavior (and the other individuals behavior) as performed in the past.

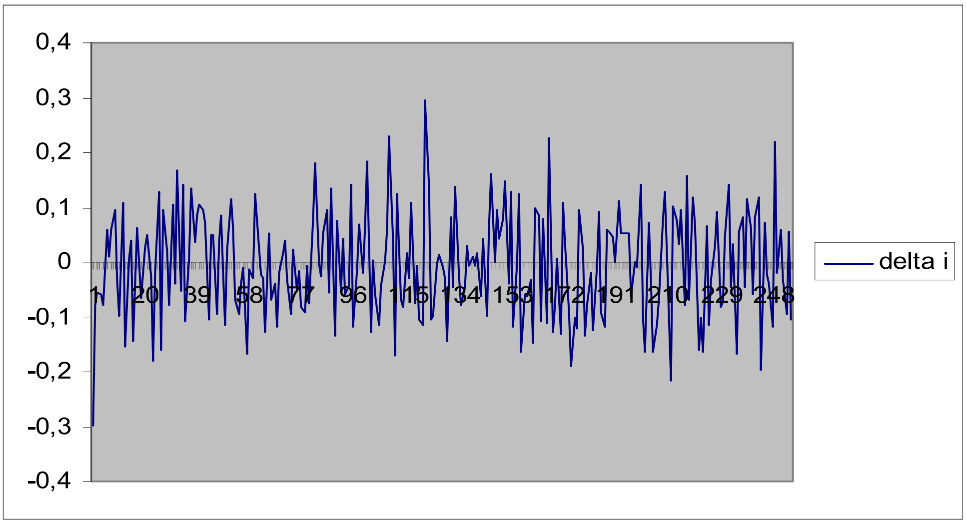
The important point of our analysis is that the heterogeneous agents / bounded rationality setup can lead to unpredictable time paths for economic variables. This was illustrated with a general example, where fitness functions regarding present and past results were assumed and also with an example concerning utility optimization and investment decisions. In each one of the cases, it became clear that it is necessary more than a bounded rationality system for the time series of variables to display strange dynamics. It is also indispensable that the two (or more) time series relating to the two (or more) behavioral rules intersect with some frequency, that is, one of the rules should not be better than the other in all moments of time because in this case heterogeneity will not be maintained and therefore one of the time series will end up by corresponding to aggregate behavior.

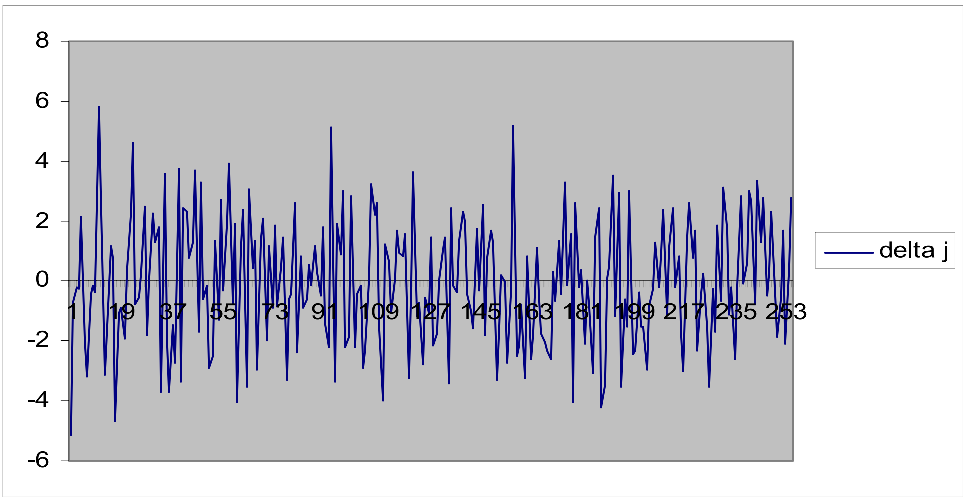
The economic example assumed in section 5 fills the previous requisite: a fixed income investment gives sometimes a high income than an investment with risk and the same expected value, but the opposite occurs with precisely the same probability. In this way, consumption growth rates that depend on the rate of return of investment will give place to utility results that also alternate in terms of performance – the utility of agents of type i (that invest in riskless assets) is sometimes higher and sometimes lower than the utility of agents of type j (that invest in risky assets). The utility results are reflected on the shares of individuals selecting one of the two investment strategies, which in turn has impact over the long run growth rate of consumption of the economy, that has to be an average of individual consumption growth paths.

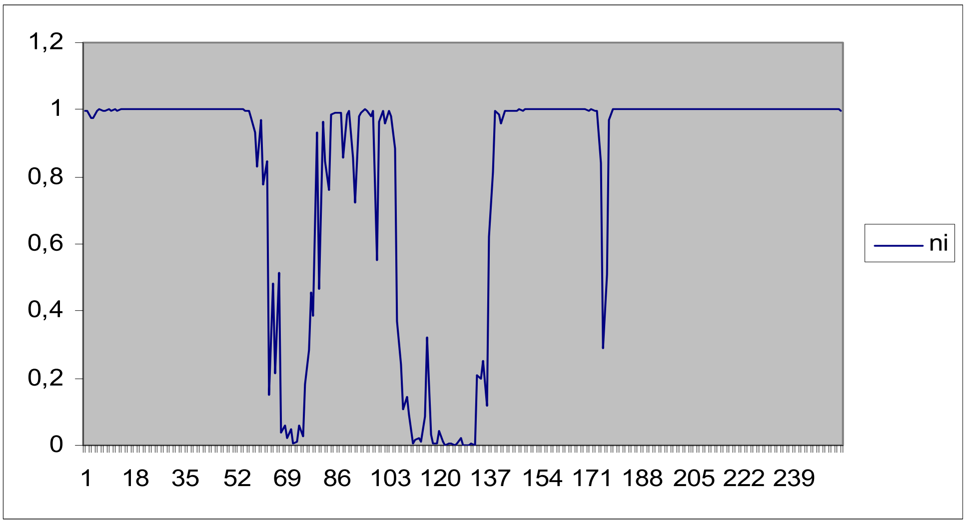
The framework of section 4 is a general framework and the setup of section 5 is meant to be an application. Many other cases where the conditions referred in the previous paragraphs are fulfilled can be considered. For instance, a model where firms can choose between two R&D strategies with uncertain results may be able to explain erratic profit paths or a setup where a world with many countries choosing between two or more trade policies can be a way to present growth paths that are impossible to predict.

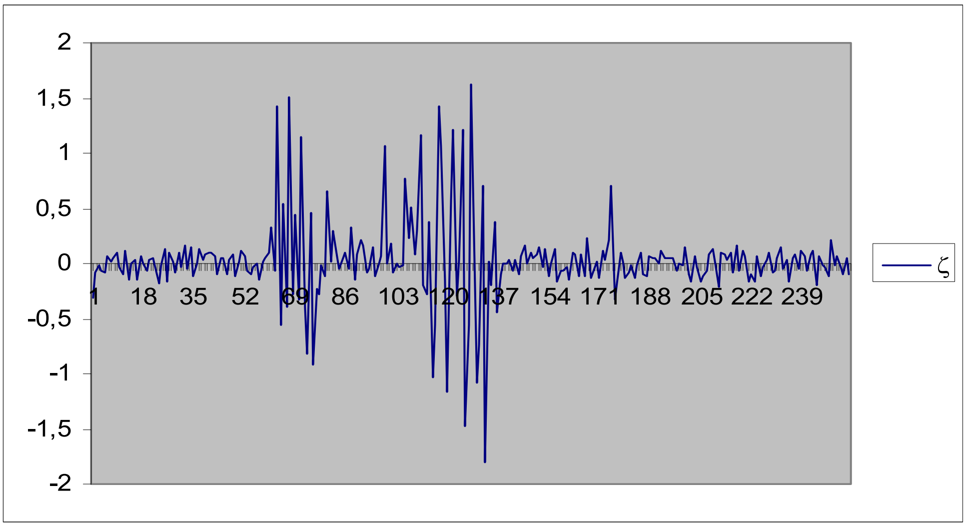
APPENDIX [I] – THE BASIC SETUP RESULTS

EXAMPLE 1





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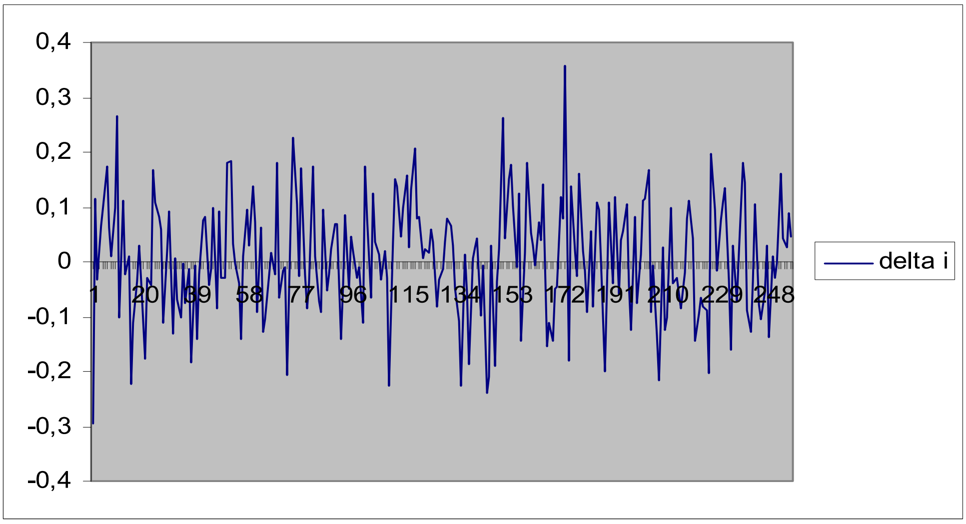
ζ

EXAMPLE 2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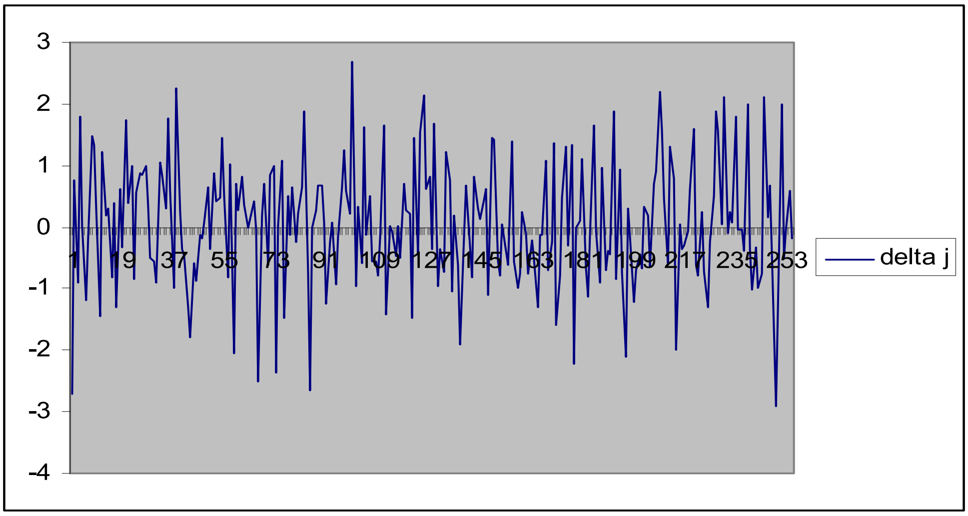
delta i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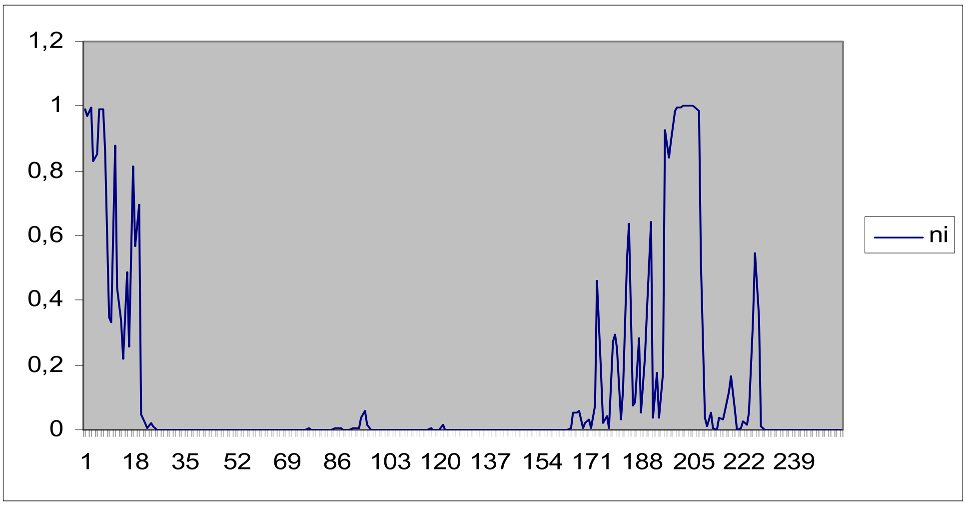
delta j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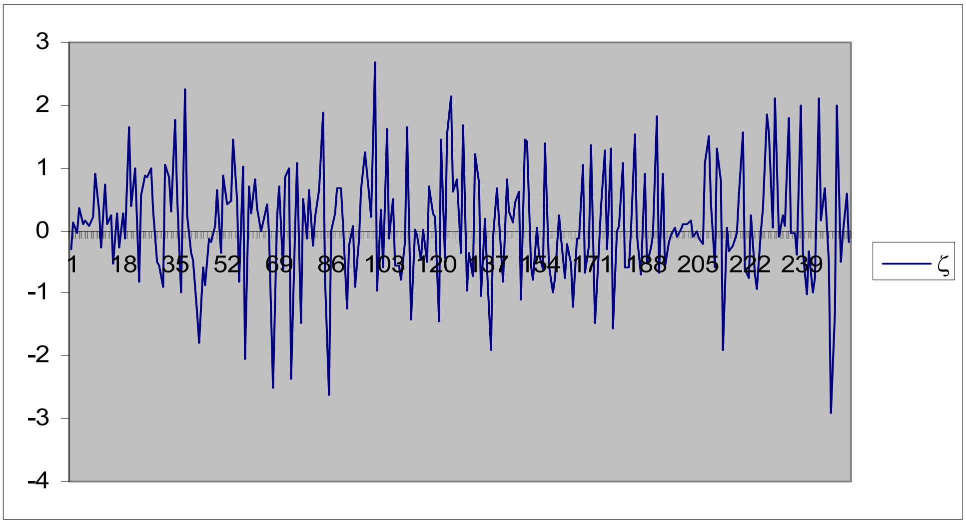
ni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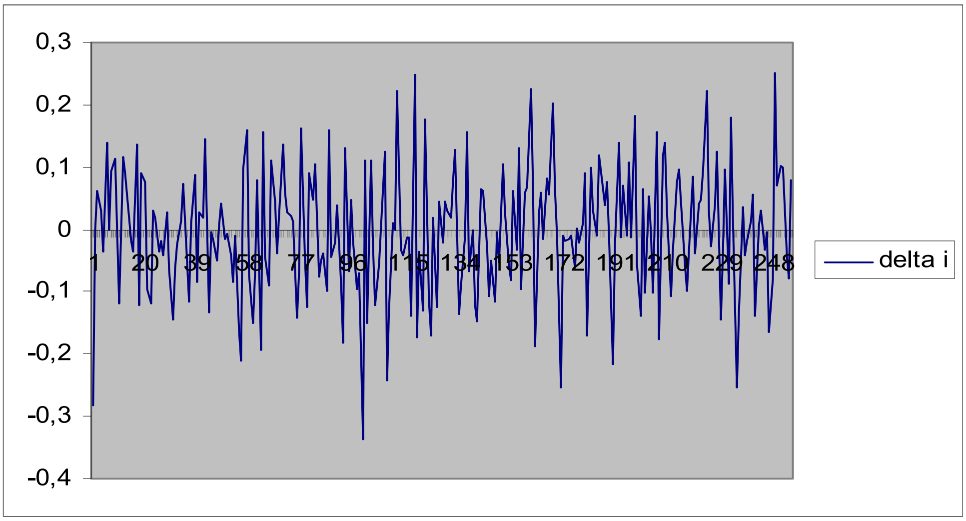
EXAMPLE 3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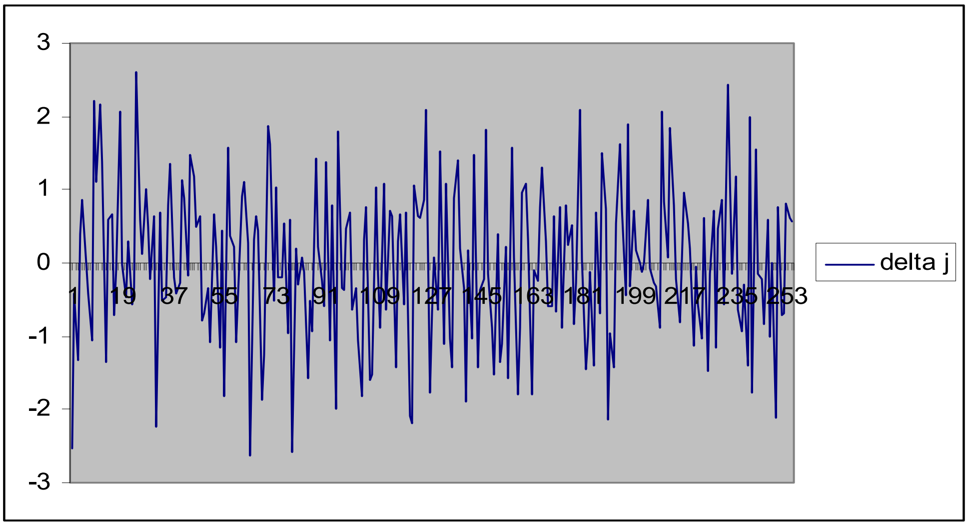
delta i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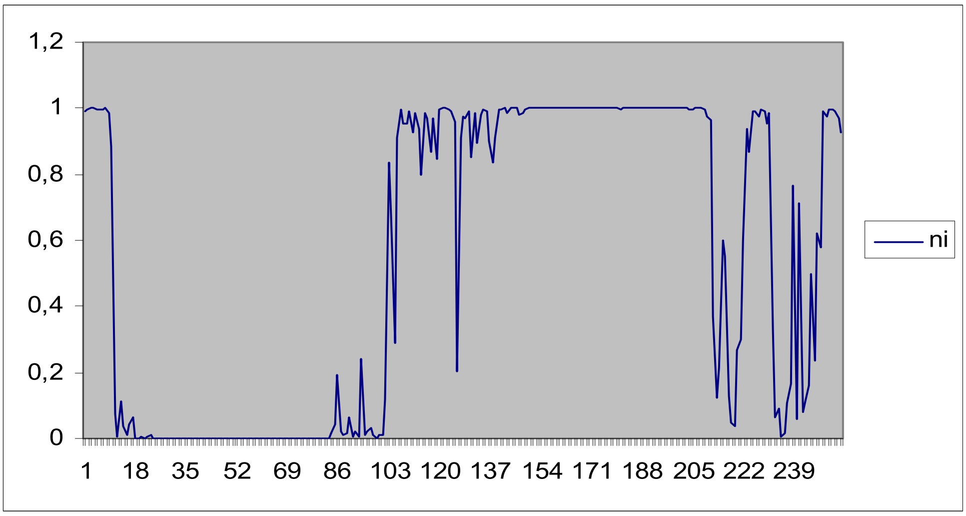
delta j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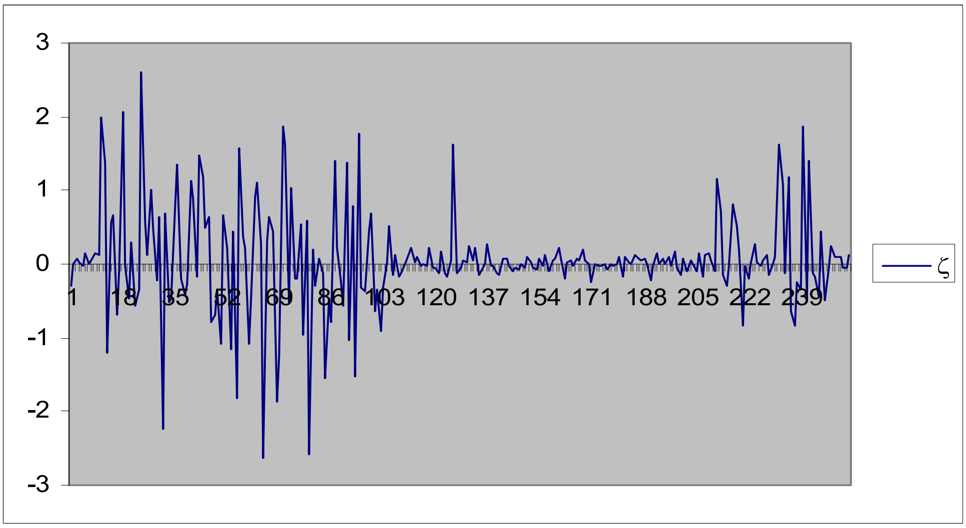
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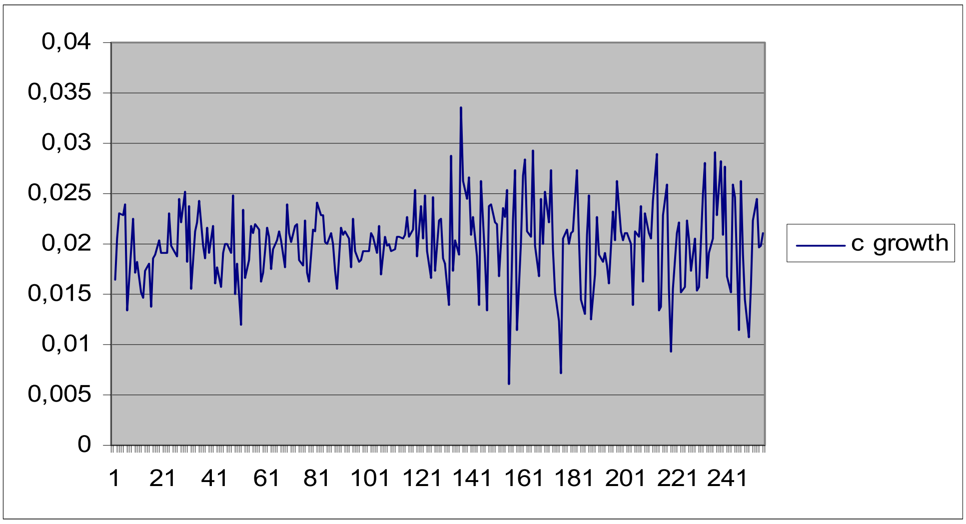
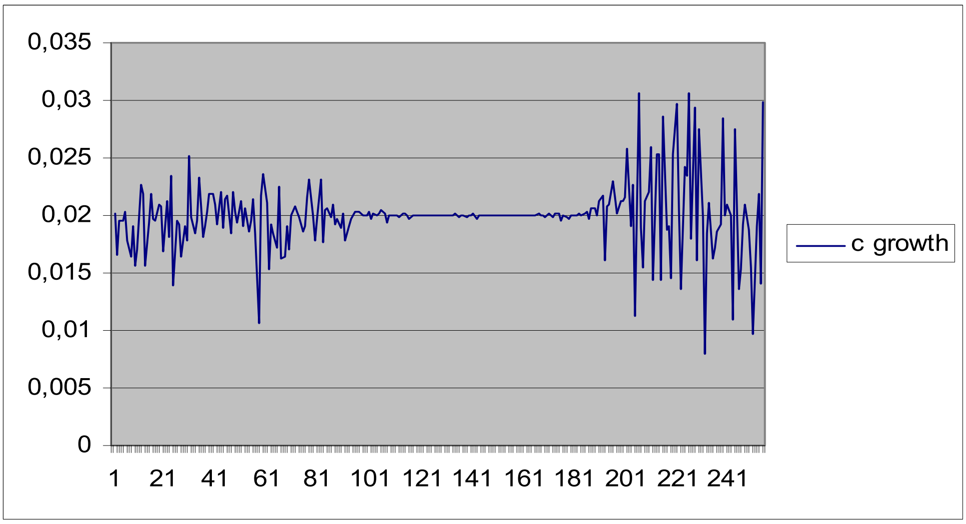
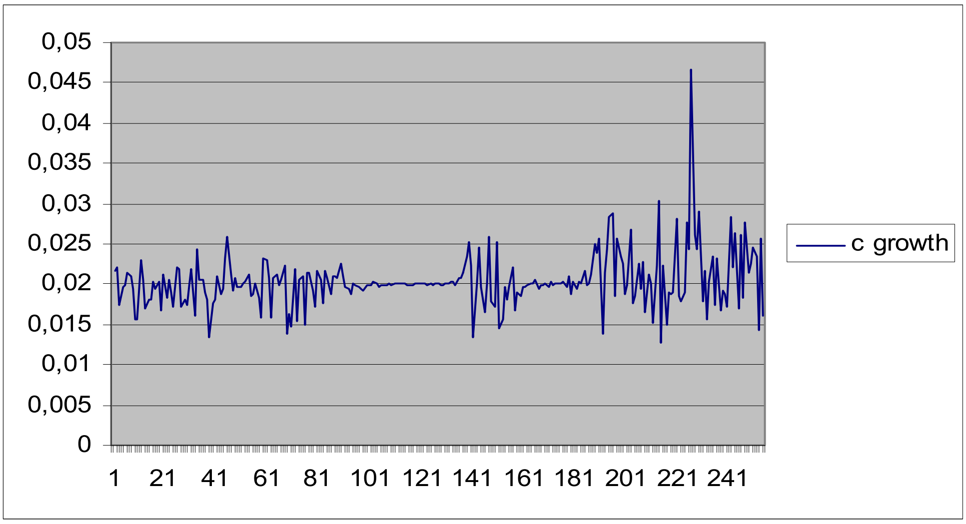
ζ

APPENDIX [II] – CONSUMPTION GROWTH RATE IN A HETEROGENEOUS AGENTS SETUP







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