Bubble-and-bust dynamics under walrasian asset pricing and heterogeneous traders *

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Abstract

Mainstream economic theory is hardly capable to explain some of the stylised facts that are normally observed in actual financial time series. Rather, phenomena like volatility clustering and excess comovement of prices have been successfully investigated in frameworks featuring heterogeneous agents and bounded rationality. Our model inherits some of the assumptions common to the Heterogeneous Agents stream of research, and develops an Agent-Based numerical simulation able to study the whole transitional price dynamics of a risky security, and the evolution of portfolio choices and wealth distribution among the traders. Adopting this methodology, we are able to show the emergence of transient bubble-and-bust dynamics, intended as sharp decoupling of the asset price from underlying fundamentals, and to replicate recent findings in financial literature about the asymptotic wealth dominance of the least-risk-averse trader, under quite general assumptions.

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1 Introduction

During the late 2000s financial crisis and the triggered Great Recession, an increasing number of people, both households and officials, felt puzzled and disappointed as no economist (with negligible exceptions, notably Roubini, 2006; Schiff and Downes, 2007) has been able to predict such catastrophe. Prevailing theoretical approach, still in use today and largely reliant on Dynamic-Stochastic General-Equilibrium (hereafter DSGE) methodology, is then put under pressure and regarded with growing scepticism. Its building blocks, related to the assumptions of *efficient* market, *representative* agent, and *rational* expectations, are progressively showing their inadequacy in explaining the economic dynamics of a day-by-day more complex and interconnected world. As a consequence their counterparts, namely market *frictions*, *heterogeneity*, and *bounded rationality*, are slowly (re-)gaining popularity in current research. Nevertheless, in a much more subtle way, these ideas have constituted the core of a largely neglected literature during the last couple of decades. In particular, the interaction of boundedly rational heterogeneous agents in economics is, to a large extent, analysed by means of two distinct but partially overlapping strands of methodology: Heterogeneous Agent Models (hereafter HAMs), and Agent-Based Models (hereafter ABMs). The first class, thoroughly surveyed by Hommes (2006), uses mathematical tools to derive strict analytical investigation of the dynamical systems driving the laws of motion of the economy. These models are indeed useful in studying the dynamics and stability properties of an economy where agents take decisions by means of heuristics or relatively simple rules of thumb. The second class, belonging to the broader area of computational economics, relies on extensive computer simulation in order to grasp the *emergence* of implicit phenomena arising from the interaction among a multitude of agents. Moreover, ABMs prove crucially useful in studying the evolution of objective variables when HAMs' dynamical systems become intractable or analytic solutions are unobtainable. Relevant literature, to which this paper is intended a contribution, is broadly surveyed in Tesfatsion and Judd (2006), and specifically, regarding financial applications, by LeBaron (2006).

We present a generic Walrasian asset-pricing model, similar to that derived in Anufriev et al. (2006), and we reproduce its relevant results in an Agent-Based computational framework. We consider a pure exchange economy with one risky assets and a riskless bond, where heterogeneous adaptive traders take investment decisions according to forecasts of future returns, based on past market observables, with constant relative risk averse (CRRA) and mean-variance attitude. The adoption of this methodology makes us able to relax some of the constraint imposed in Anufriev et al. (2012), such as the assumption of procedurally rational equilibrium; this allows to account for the whole out-of-equilibrium transitional dynamics of the system, and not just the point it asymptotically converges to. We show that such dynamics can turn out astoundingly rich, even under quite general assumptions and reasonable parameters value.

Next **Section** briefly reviews the current mainstream approach to financial markets theory and surveys some of the empirical stylised that actual financial time series usually exhibit; **Section 3** sketches the model from which we develop, in **Section 4**, the Agent-Based numerical setup and discuss the associated findings; finally, **Section 5** concludes and suggests further conceivable improvements.

2 Relevant stylized facts in financial time series

The last global financial crisis has revealed the need to fundamentally rethink the way financial systems work and related research is carried out. The implicit view behind standard models is that markets and economies are inherently stable and that they get off-track only temporarily. The majority of economists thus failed to warn policy-makers about the threatening systemic crisis and ignored the work of those who did. *"The confinement of macroeconomics to models of stable states that are perturbed by limited external shocks and that neglect the intrinsic recurrent boom-and-bust dynamics of our economic system is remarkable"* (Colander et al., 2009). Regarding financial economics, this idea has been basically translated into a number of highly questionable assumptions, still regarded as normal practice nowadays, including market completeness, behaviour and expectation rationality, agent representativeness, and market efficiency. From the empirical point of view, none of the above assumptions is really validated. Actual financial time series often exhibit challenging statistical properties, many of which still cannot be properly explained by incumbent theory (see Cont, 2001).

Volatility clustering. Volatility displays auto-correlation over time, capturing the fact that high volatility events tend to cluster together, resulting in persistence of the amplitudes of price changes. Mandelbrot (1963) recognizes that "*large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes*". While the linear auto-correlation function of price changes tends to rapidly decay in a few minutes, especially in high-liquidity markets (this being often cited as supporting the EMH, see Fama, 1991), it is not enough to imply the independence of the increments; auto-correlation is significantly present in nonlinear functions, such as the absolute value or the square of returns (Ding et al., 1993), undermining a key assumption of the random-walk model.

Excess volatility. The variability of asset prices is usually not justifiable by variation in related economic fundamentals. The occurrence of large returns (in absolute value) is unexplainable by new information coming available. Cutler et al. (1989) remark the difficulty of explaining as much as half of the variance in stock prices on the basis of publicly available news bearing on fundamental values; a clear example is the sharp drop in stock prices occurred on October 19th, 1987 in the complete absence of news about fundamentals. Another striking empirical observation has been the strong appreciation followed by a strong depreciation of the dollar in the mid eighties, which seemed to be unrelated to economic fundamentals (see Frankel and Froot, 1986). Moreover volatility has been showed to systematically exceed that justifiable by fundamentals (see Shiller, 1981, 1989 and LeRoy and Porter, 1981).

Excess covariance. Classical finance models share the prediction that stock prices shall move together only in response to common variations in fundamentals. However this is not always the case. Pindyck and Rotemberg (1993) show that stock returns of companies in unrelated businesses co-move significantly more that can be explained by common variations in discount rates. On the other hand, Froot and Dabora (1999) find that returns of *siamese-twin companies*, whose cash-flows are perfectly correlated, lay far from perfectly co-moving. Additional results show that covariance and correlation across asset returns change over time and according to the business cycle, with average correlation being higher during bad times (see Ribeiro and Veronesi, 2002).

Heavy tails. Distribution of returns tends to show significant leptokurtosis (see Guillaume et al., 1997 and again Mandelbrot, 1963 and Ding et al., 1993), likely displaying a power-law or Pareto-like tail (Cont, 2001). This finding is, alone, able to cast doubts on the validity of classical portfolio theory, the Black-Scholes-Merton option pricing model (see Black and Scholes, 1973 and Merton, 1973) or the RiskMetrics™ variance-covariance approach to Value at Risk, all relying on *normality* assumption of returns.

While the aforementioned phenomena have been already investigated by economic literature, financial bubbles are not generally included among the financial time series stylized facts, and little effort has been put in studying their formation. Quoting Stiglitz (1990), "*if the reason that the price is high today is only because investors believe that the selling price will be high tomorrow — when 'fundamental' factors do not seem to justify such a price — then a bubble exists*". Clearly, the existence of such bubbles stands quite at odds with the efficient market hypothesis (Fama, 1970), asserting that all the price swings continuously observable in financial markets are the mere instantaneous and unbiased adjustments towards the equilibrium, triggered by new public available information or, put differently, to changing fundamentals. The pervasive presence of leptokurtosis then challenges the EMH prescription about the impossibility of abnormal capital gains. To sum up, mainstream economists believe the sources of market instability have to necessarily be external to the market, and come as exogenous unpredictable shocks. Occurrences such as financial bubbles are then conveniently believed to be the exception, rather than the rule, irrespectively of their devastating consequences.

On the very opposite side, the majority of the theoretical literature able to replicate to some extent the aforementioned stylized facts departs, to different degrees, from the canonical EMH model and falls within the two partially overlapping classes of *Heterogeneous Agents Models* and *Agent-Based Models*. Under both perspectives, *agents* (and their behaviour) stand at the core of the study, and population *heterogeneity* is a key common feature (see also Kirman, 1992; Levy and Levy, 1996). Belonging to the first class, Anufriev and Bottazzi (2010), Anufriev et al. (2006), and Anufriev and Dindo (2010) introduce and exploit tools such as the *equilibrium market line* and the concept of *procedurally rational equilibrium* to provide asymptotic dominance results of different trading strategies in terms of relative performances, and subsequent wealth-driven selection. Besides the great achievements reached through the HAM methodology, a couple of intrinsic features are, in our opinion, rather binding. The need for analytical tractability often leads to restrictive simplifying assumptions, even when phenomena are *complex* by their very nature, and preferably require such complexity to be studied as is. Moreover, most of the results obtained under this approach focus on the *asymptotic* properties of the models. On the contrary, the methodological perspective that we propose here, belonging to the second class, takes the full sample paths as *transients* of the evolving dynamics. Remarkable contributions under this methodology include the still state-of-the-art Santa Fe Artificial Stock Market (see Palmer et al., 1994; Arthur et al., 1997; LeBaron et al., 1999) which proposes a theory of asset pricing in an artificial economy where a population of heterogeneous traders continually explore new expectational models, and confirm or discard them according to their performances, therefore endogenising individual beliefs. Another masterpiece, contemporary to the Santa Fe work, is the Levy-Levy-Solomon Microscopic Simulation Model (see Levy et al., 1994, 1995, 1996): in making their optimal diversification choice between a risky and a riskless asset, traders employ the ex-post distribution of returns as an estimate of the ex-ante distribution, keeping track of the last 10 historical observations. Traders are homogeneously initialised in terms of behavioural parameters, expectations formation mechanism, and wealth distribution. The only source of heterogeneity is a normally-distributed random shock that is added to individual optimal investment, whose variance regulates the 'temperature' of the system. Market suffers discontinuities (e.g. booms and crashes) especially when the shock variance is low, while, somewhat counterintuitively, cycles become milder and crashes much smaller if the 'heater is turned on'. This model bears close similarities with the one we propose in the next **Section**, although, after Zschischang and Lux (2001), we introduce agent heterogeneity by means of behavioural parameters such as the magnitude of risk aversion and of the memory decay factor in expectations formation. The model which follows, starts from a setting common to a number of HAMs and then develops a computational ABM to investigate the whole dynamics of the system under different parametrisations.

3 The model

As already mentioned, our model builds upon the basis put forth by Anufriev et al. (2012, 2006) and is developed in order to explicitly account for the whole transitional dynamics of market variables, such as prices, returns, composition of the market-portfolio, and individual ones, such as the evolution of traders' wealth-shares. Heterogeneity is introduced both in traders' attitude towards risk, and in the way they form beliefs about future states-of-the-world. The central purpose is to show that, by means of marginal departures from the mainstream assumptions surveyed above, the dynamic properties of relevant variables may dramatically change.

Consider a pure-exchange economy populated by a set of agents indexed by $\mathscr{N} = \{1, \dots, n, \dots, N\}$, where a set of risky long-lived securities, indexed by $\mathscr{L} = \{1, \dots, \ell, \dots, L\}$, and a riskless bond, are traded in discrete time $t \in \mathscr{T}$. Risky securities pay a random dividend d_t^ℓ at the end of each period. Before trade at time t starts, each agent n chooses a fraction $x_{n,t}^\ell$ of the wealth he possesses, to be invested in security ℓ. The decision is made according to currently available information, namely past realized prices and dividends, and coherently with a constant relative risk averse attitude. The residual part of wealth not invested in the L risky securities, is risklessly lent, by means of bond purchase, at a constant exogenous rate of return $r_f > 0$. The amount of circulating shares of the risky securities is constant, while the supply of bond is inelastic. Individual demands are then aggregated by a Walrasian auctioneer who announces the end-of-period price vector $\boldsymbol{p}_t \in \mathbb{R}^L$ obtained by setting the aggregate excess-demand equal to zero. Dividends $(d_t^1, \dots, d_t^L)$ and (ex-dividend) prices $(p_t^1, \dots, p_t^L)$ of the risky securities are expressed in terms of the bond's price, the latter serving as the *numéraire*, conventionally normalized to 1 in every period. The economy runs through a series of *temporary equilibria* (see Grandmont, 1985) where market clearing condition is satisfied. Next **subsection** presents the way traders behave and the determinants of their investment functions; **Sections 3.2** and **3.3** define, accordingly, the evolution of individual and aggregate wealth, and derive asset pricing; **Section 3.4** specifies the expected utility maximisation problem each trader solves, and the estimators employed in forming expectations about future states of nature; finally, **Section 3.5** defines the whole dynamical system for an economy with heterogeneous traders.

3.1 Agent behaviour

An individual investment decision is a vector $\boldsymbol{x}_{n,t} \in \mathbb{R}^L$ of fractions of wealth the trader is willing to invest in each risky asset ℓ at time t. A restriction we impose here is that investment decision takes place before trade starts at each round, thus agent information set $\mathcal{I}_{n,t}$ is restricted to include only past information on the realisation of prices and dividends:

$$\mathcal{I}_{n,t} = \{p^1_\tau, \dots, p^L_\tau;\ d^1_\tau, \dots, d^L_\tau \mid \tau < t\} \qquad \forall n \in \mathscr{N},\ \forall t \in \mathscr{T} \tag{3.1}$$

At every time step, all agents costlessly acquire all the relevant information about the time series of prices and dividends up to the last trade session $t-1$; since $\mathcal{I}_{n,t}$ is unbiased common knowledge we can drop the subscript n. Current values of prices and dividends, $\boldsymbol{p}_t$ and $\boldsymbol{d}_t$, being determined during time t trade, cannot belong to $\mathcal{I}_t$.

Investment strategy is the image of a trader-specific investment function:

$$f_n : \mathbb{R}^{\tau \times L} \longrightarrow \mathbb{R}^L \qquad \text{with } \tau < t \in \mathscr{T} \tag{3.2}$$

f_n deterministically maps the information set $\mathcal{I}_t$ available at time t into a portfolio of risky assets $\boldsymbol{x}_{n,t}$. In a dynamical context the individual investment fractions are, in general, changing as new information becomes available. Since the investment decision of agent n at time t is completely described by the vector of investment fractions $\boldsymbol{x}_{n,t}$, agents adopting the same strategy can be considered, without loss of generality, as a single agent, thus we assume there are N distinct investment functions in the economy, each associated to a level of wealth $W_{n,t}$. According to the investment function defined in (3.2), individual demand at time t reads

$$Z^\ell_{n,t} = \frac{x^\ell_{n,t} \cdot W_{n,t}}{p^\ell_t} \qquad \forall \ell \in \mathscr{L} \tag{3.3}$$

i.e. it equals the amount of wealth allocated for investment in risky asset ℓ in monetary terms, divided by the still unknown prevailing price of the asset. A demand function like (3.3) along with the independence of $x^\ell_{n,t}$ on both $W_{n,t}$ and p^ℓ_t implies a specific dependence of agents' demand on wealth and prices and amounts to assuming that agents have constant relative risk averse (hereafter CRRA) attitude. This assumption is common to a number of studies in the HAMs literature (see e.g. Anufriev et al., 2012, 2006; Chiarella and He, 2001; Levy et al., 2000) while other works adopt a Constant Absolute Risk Averse (CARA) attitude, with demand (rather than optimal wealth fractions) not depending on current wealth (see e.g. Brock and Hommes, 1998). Following Levy (1994), we believe CRRA specification to closer mimic the way financial decisions are taken in the real world, where portfolios are often designed as fractions of wealth to be split across different securities. Finally, notice that the model does not include any consumption in agent behaviour, hence it represents a pure-exchange economy where traders decisions are reasonably driven by expectations about future wealth.

3.2 Agent wealth

Agent wealth at time t consists of the current market value of his portfolio:

$$W_{n,t} = A_{n,t} \cdot \boldsymbol{p}_t + B_{n,t} \tag{3.4}$$

where $A_{n,t} \in \mathbb{R}^L$ denotes the (row) vector of the amount of risky assets held by agent n at time t after market clearing, $B_{n,t} \in \mathbb{R}$ denotes the corresponding holding of the bond (whose price is unitary), and the inner product is understood. The inter-temporal evolution of individual wealth thus develops according to:

$$\begin{aligned} W_{n,t} &= W_{n,t-1} \cdot \left(1 - \sum_{\ell=1}^{L} x_{n,t-1}^{\ell}\right) \cdot (1 + r_f) + W_{n,t-1} \cdot \sum_{\ell=1}^{L} x_{n,t-1}^{\ell} \cdot \frac{p_t^{\ell} + d_t^{\ell}}{p_{t-1}^{\ell}} \\ &= W_{n,t-1} \cdot \left[x_{n,t-1}^{0} \cdot (1 + r_f) + \sum_{\ell=1}^{L} x_{n,t-1}^{\ell} \cdot \left(1 + \rho_t^{\ell} + e_t^{\ell}\right) \right] \end{aligned} \tag{3.5}$$

where $x_{n,t}^{0} = 1 - \sum_{\ell=1}^{L} x_{n,t}^{\ell}$ denotes the fraction of wealth used for the bond purchase, i.e. the complement to 1 of the risky holdings. This accounting relation clearly shows the sources of wealth growth, namely capital gain (i.e. the price rate of return $\rho_t^{\ell} \stackrel{\text{def}}{=} \frac{p_t^{\ell}}{p_{t-1}^{\ell}} - 1$) and the dividend yield $e_t^{\ell} \stackrel{\text{def}}{=} \frac{d_t^{\ell}}{p_{t-1}^{\ell}}$ for the share of wealth risky invested, and the risk-free rate of return r_f for the share of wealth risklessly lent.

We now state a couple of definitions that will prove useful in the following subsections:

◇ **Aggregate wealth**: is the sum of all individual wealth levels at a time instant t:

$$W_t \stackrel{\text{def}}{=} \sum_{n=1}^{N} W_{n,t} \tag{3.6}$$

◇ **Individual wealth-shares**: are the ratios of each individual wealth out of aggregate wealth at a time instant t:

$$\varphi_{n,t} \stackrel{\text{def}}{=} \frac{W_{n,t}}{W_t} \qquad \forall n \in \mathcal{N} \tag{3.7}$$

◇ **Market portfolio**: is the wealth-weighted sum of individual portfolios:

$$\boldsymbol{x}_t \stackrel{\text{def}}{=} \sum_{n=1}^{N} \boldsymbol{x}_{n,t} \cdot \varphi_{n,t} \tag{3.8}$$

3.3 Asset pricing

Starting with the aforementioned individual demand functions (3.3), we can normalize, without loss of generality, the supply of each risky asset to 1, such that the pricing condition comes from the equilibrium relation:

$$\sum_{n=1}^{N} \boldsymbol{Z}_{n,t} = \boldsymbol{1} \tag{3.9}$$

where $\mathbf{1} = [1, \ldots, 1]^T \in \mathbb{R}^L$. Solving for price yields

$$p_t^\ell = W_t \cdot x_t^\ell \qquad \forall \ell \in \mathscr{L} \tag{3.10}$$

where W_t is the aggregate wealth at time t and x_t^ℓ is the ℓ-th component of the market portfolio. In (3.10) asset prices still appear both in the LHS and the RHS of the equation, as determinants of the level of wealth (recall eq. 3.5). It is then possible to show that:

Proposition 1. *If short positions are not allowed, i.e.*

$$x_{n,t}^\ell \in (0,1) \qquad \forall n \in \mathscr{N},\ \forall \ell \in \mathscr{L},\ \forall t \in \mathscr{T} \tag{3.11}$$

then prevailing prices exist, are unique and strictly positive. Moreover, it holds:

$$W_t = W_{t-1} \cdot \frac{x_{t-1}^0 \cdot (1 + r_f) \cdot \sum_{\ell=1}^L x_{n,t-1}^\ell \cdot e_t^\ell}{x_t^0} \tag{3.12}$$

$$p_t^\ell = p_{t-1}^\ell \cdot \frac{x_t^\ell}{x_{t-1}^\ell} \cdot \frac{x_{t-1}^0 \cdot (1 + r_f) + \sum_{\ell=1}^L x_{t-1}^\ell \cdot e_t^\ell}{x_t^0} \tag{3.13}$$

Proof. See **Appendix A**. □

Notice that condition (3.11) is only sufficient and in general can be relaxed for individuals as long as it holds at the aggregate level, i.e. as long as $x_t^\ell \in (0,1)$, $\forall \ell \in \mathscr{L}, \forall t \in \mathscr{T}$. Nonetheless, at the moment, we make the following assumption:

Assumption 1. *No trader can take short position in any asset, i.e. the image of traders' investment functions is restricted to the interior of the L-dimensional unit simplex*

$$f_n : \mathbb{R}^{\tau \times L} \longrightarrow \mathrm{Int}\left(\triangle^L\right) \tag{3.14}$$

that is, both $x_{n,t}^\ell$ and $x_{n,t}^0$ satisfy condition (3.11), $\forall \ell \in \mathscr{L}$, $\forall n \in \mathscr{N}$, $\forall t \in \mathscr{T}$.

3.4 Expectations and the investment function

At every time step, every agent solves an optimisation problem of the form:

$$\max_{\boldsymbol{x}_t} \mathbb{E}\left[U(W_t)\right] \quad \text{s.t.} \quad W_t = W_{t-1} \cdot \left[x_{t-1}^0 \cdot (1 + r_f) + \boldsymbol{x}_{t-1} \cdot (1 + \boldsymbol{\rho}_t + \boldsymbol{e}_t)\right] \tag{3.15}$$

For purely notational convenience the subscript n is temporarily dropped for the remainder of this section. $U(W_t)$ represents the trader's utility function of wealth. In compliance with CRRA attitude, its generic form reads:

$$U(W_t) = \frac{W_t^{1-\gamma} - 1}{1 - \gamma} \tag{3.16}$$

where $\gamma > 0$ denotes the individual relative risk-aversion coefficient. Since the solution of maximisation (3.15) is independent on the current level of wealth, we need to formally model the way agents form their expectations over future returns. Here we make the following additional assumption:

Assumption 2. *The dividend yield e_t^ℓ is drawn at each time step from a L-dimensional probability distribution with mean $\bar{e}$ and covariance matrix $\mathbf{\Sigma}$.*

Coherently with Anufriev et al. (2006) and Anufriev et al. (2012), we assume the trader forms expectations on future returns of risky assets according to a smooth function of their Exponentially Weighted Moving Average (hereafter EWMA) estimates:

$$\hat{\rho}_t^\ell = \lambda \cdot \sum_{\tau=0}^{\infty} (1-\lambda)^\tau \cdot \rho_{t-\tau-1}^\ell \tag{3.17}$$

$$\hat{\sigma}_{\rho,t}^{\ell,h} = \lambda \cdot \sum_{\tau=0}^{\infty} (1-\lambda)^\tau \cdot \left[\rho_{t-\tau-1}^\ell - \hat{\rho}_{t-\tau-1}^\ell\right] \cdot \left[\rho_{t-\tau-1}^h - \hat{\rho}_{t-\tau-1}^h\right] \tag{3.18}$$

where $\hat{\rho}_t^\ell$ and $\hat{\sigma}_{\rho t}^{\ell h}$ denote the trader's expectations about the return of asset ℓ and its (co-)variance (with respect to asset h) at time t, and $\lambda \in (0,1)$ is a parameter governing the decay of relative weights between recent and remote observation of realized returns. The EWMA estimators above also admit recursive definition:

$$\hat{\rho}_t^\ell = (1-\lambda) \cdot \hat{\rho}_{t-1}^\ell + \lambda \cdot \rho_{t-1}^\ell \tag{3.19}$$

$$\hat{\sigma}_{\rho,t}^{\ell,h} = (1-\lambda) \cdot \hat{\sigma}_{\rho,t-1}^{\ell,h} + \lambda \cdot (1-\lambda)^2 \cdot \left[\rho_{t-1}^\ell - \hat{\rho}_{t-1}^\ell\right] \cdot \left[\rho_{t-1}^h - \hat{\rho}_{t-1}^h\right] \tag{3.20}$$

Applying the same mean-variance approximation employed in Chiarella and He (2001) and Anufriev et al. (2012), the individual investment function holds:

$$x_t^\ell = f^\ell\left(\mathcal{I}_t\right) = \frac{1}{\gamma} \cdot \mathbb{C}_t^{-1} \cdot \left[\mathbb{E}_t - r_f \cdot \mathbf{1}\right] \tag{3.21}$$

where $\mathbb{E}_t$ and $\mathbb{C}_t^{-1}$ are, respectively, the vector of expected total returns and the inverse of the expected variance-covariance matrix, whose elements read:

$$\mathbb{E}_t^\ell = \bar{e}^\ell + d \cdot \hat{\rho}_t^\ell \tag{3.22}$$

$$\mathbb{C}_t^{\ell,h} = \sigma_e^{\ell,h} + \hat{\sigma}_{\rho,t}^{\ell,h} \tag{3.23}$$

The d coefficient, introduced by Chiarella and He (2001), characterizes the relation of present investment choice to past market dynamics. This parameter distinguishes between different stylized types of trading behaviour: if $d = 0$ the trader acts as a *fundamentalist* since his investment choice is unaffected by past return realisations; if $d \neq 0$ the agents is a *chartist*, specifically a *trend-chaser* for $d > 0$ and a *trend-contrarian* for $d < 0$. Higher values of past returns lead to riskier investment choices for trend followers and to 'safer' investment choices for contrarians. Following **Assumption 2** the term $\sigma_e^{\ell,h}$ is null, as we assume *i.i.d.* dividend process, hence showing no cross-correlation between different risky assets.

3.5 The economy with heterogeneous agents

Starting from the relations provided in the preceding subsections, we are now able to derive the overall dynamics of an economy populated by N heterogeneous trader, which is described by a $[N(3L+1)-1]$-dimensional system of first-order difference equations: the first $N \cdot L$ equations are the individual investment functions (3.21), whose arguments are the EWMA estimates $\mathbb{E}_{n,t}$ and $\mathbb{C}_{n,t}$ of future price returns and their variance; these estimates, in turns, correspond to the second and third $(N \cdot L)$-size sets of equations. Last, there remain the equations describing the evolution of individual wealth(-shares) according to (3.5), which count $N-1$ as, by definition, last-trader's wealth-share can be computed complementarily, as $\varphi_{N,t} = 1 - \sum_{n=1}^{N-1} \varphi_{n,t}$. Rather than employing analytical methods to study this high dimensional dynamical system (refer to Anufriev et al. (2006) for a deterministic skeleton analysis of the system for $L = 1$ and to Anufriev et al. (2012) for a procedurally rational equilibrium analysis of the system for $L > 1$), we perform, in the next **Section**, an agent-based numerical simulation, in order to account for the *step-by-step* evolution of the relevant dynamics.

4 Simulation and results

Starting from the financial HAM sketched in the previous **Section**, we are now able to perform numerical simulations[1] under different scenarios. Here we investigate the case for $L = 1$, that is, with only one risky asset (that can be nevertheless intended as an index of risky assets itself) and we focus on the dynamics of its price (remember the bond acts as numéraire), the evolution of relative wealth-shares of the traders, and the overall riskiness of the market portfolio. The choice of initial conditions is deliberately arbitrary and these need no calibration on any specific financial dataset, as our simulation is not intended to exactly fit any existing time series. We are able to show two distinct results: the first, in **Section 4.1**, concerns the selection mechanism the market operates, and is able, to a good extent, to replicate the findings of Anufriev et al. (2006) about the asymptotic dominance of different trading strategies; the second, in **Section 4.2**, focusses on the *rich* and anything but linear transitional dynamics of the price series in its adjustment towards the equilibrium.

4.1 Market selection and survival patterns

The first simulation we carry out focusses on the evolution of individual wealth-shares $\varphi_{n,t}$ in order to study the way market selection occurs among different trading strategies. We sketch the following

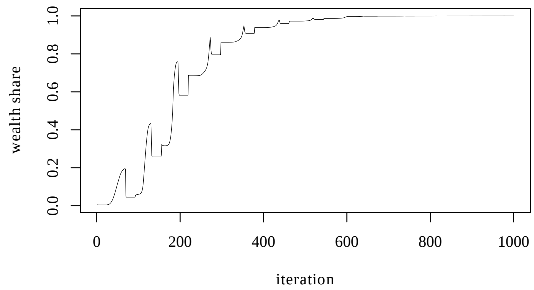
Definition 1. *A trader n is said to* "survive" *the economy if his long-run wealth-share is significantly non-zero, i.e. if $\lim_{t\to\infty} \varphi_{n,t} > 0$. A trader n is said to* "dominate" *the economy if his long-run wealth-share is significantly close to 1, i.e. if $\lim_{t\to\infty} \varphi_{n,t} = 1$.*

Anufriev et al. (2006) show that, under **Assumption 1**, two types of equilibria are possible: one featuring a single survivor (who therefore dominates the economy), and one with multiple survivors, the first being a particular case of the second. Studying the stability conditions for a generic single-survivor equilibrium, they are also able to show that, in those equilibria where the condition $r^* > -\bar{e}$ is satisfied, i.e. where the overall wealth of the economy grows, the survivor must be the most 'aggressive' trader, say, the one who, among the others, invests the highest share of wealth $x_{n,t}$ in the risky security (or, equivalently, the lowest share of wealth $x^0_{n,t}$ in the bond). Our first simulation is devoted to the validation of this result. Initialisation parameters are listed in Table 1 (index ℓ is dropped

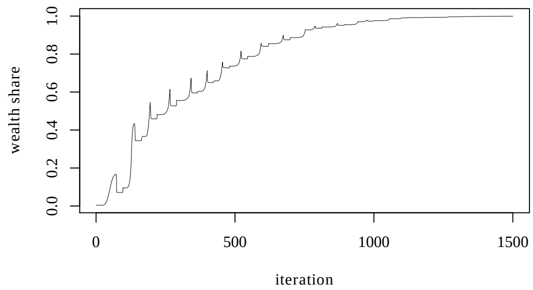
[1] Algorithms are coded in Java™ and implemented with openJDK v. 7u60; random number generation relies upon colt libraries v. 1.2.0.

Description	Value
Initial population size	$N = 200$
Number of risky assets	$L = 1$
Static population	`true`
Riskless rate of return	$r_f = 0.02$
γ distribution	$\gamma_n \sim \mathcal{U}(1.0, 1000.0)$
λ distribution	$\lambda_n = 0.1,\ \forall n \in \mathcal{N}$
d distribution	$d_n = 1.0,\ \forall n \in \mathcal{N}$
Initial wealth endowment	$W_{n,0} = 50.0,\ \forall n \in \mathcal{N}$
Yield mean	$\bar{e} = 0.04$
Yield variance	$\sigma_e^2 = 1.0\text{e-}4$
Yield realisation distribution	$e_t \sim \mathcal{N}(\bar{e}, \sigma_e^2)$
Initial risky asset price level	$p_0 = 0.1$
x_n^ℓ admissible interval	$x_{n,t} \in [0.01, 0.99],\ \forall n \in \mathcal{N}, \forall t \in \mathcal{T}$

Table 1: Parameters and initial conditions.



(a) A typical run.



(b) Another typical run.

Figure 1: Evolution of the wealth-share for the least-risk-averse trader.

for notational convenience). We keep a steady population of 200 traders which differ one to each other in their risk aversion coefficient γ. Memory decay factor λ of expectation formation and overall sensibility coefficient d to the EWMA estimates are kept constant across the population in order to focus on the role of differences in the magnitude of risk aversion. If both $\lambda_n = \lambda$ and $d_n = d$, $\forall n \in \mathcal{N}$, there is a monotone relation between the risk aversion coefficient and the riskiness of the optimal solution of the UMP: *cæteris paribus*, the lower γ_n the higher $x_{n,t}$ (see eq. 3.21). It is important to notice here that at the beginning of the run, wealth is evenly (*uniformly*) distributed among all traders, so that whatever inequality may arise as trading takes place must be induced only by differences in behavioural attitudes, as no trader starts trading with any relative advantage in terms of purchasing power. At time $t = 0$ each trader faces the market holding a riskless portfolio worth $W_{n,0}$, entirely consisting of bond shares. The risky security is exogenously priced p_0, and carries no information on past prices and dividend (in other words, the first iteration can be regarded as an IPO). Figure 1 shows the evolution of the individual wealth-share for the trader featuring the lowest value of γ, in two typical runs. As suggested, with time going by, the least-risk-averse trader tends to increase his own wealth

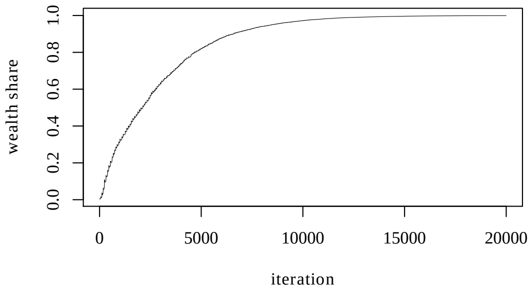


Figure 2: Evolution of the wealth-share for the least-risk-averse trader; narrower γ support: $\gamma_n \sim \mathcal{U}(100.0, 1000.0)$

more than any other, and eventually the whole aggregate wealth will concentrate in his hands, making him a lone survivor, necessarily dominating the economy. Even without a clear-cut definition of time horizon, it is straightforward to notice that the adjustment mechanism needs not be monotone in the short run, where other traders (who will not survive at the equilibrium) may locally perform better; long run graph (or its moving average) nonetheless looks roughly monotone. Although the speed of adjustment may vary for purely stochastic reasons, e.g. it takes about 50% longer for the trader to dominate the economy for a given confidence level (0.001) in the second run with respect to the first (circa 1,500 vs. 1,000 iterations), it is highly influenced by the width of the support parameter γ_n is drawn from. Figure 2 shows the same type of graph as before, but for a narrower support of coefficient γ. With respect to Table 1, the only difference here is that $\gamma_n \sim \mathcal{U}(100.0, 1000.0)$. It is immediate to notice that it now takes a considerably larger amount of time (circa 19,000 iterations) for the 'lucky' trader to wipe all the other agents out of the economy, at the same confidence level. A direct consequence of the dominance condition sketched above is that, in order to have a multiple-survivor equilibrium, all investment shares $x_{n,t}$ for those n's belonging to the survivors' subset of $\mathscr{N}$ have to be identical one to each other. This implies that an economy composed of N heterogeneous traders with randomly defined investment functions (e.g. with at least one parameter appearing, directly or indirectly, in the investment function (3.21) drawn from a continuous non-degenerate distribution) has probability zero of displaying an equilibrium with more than one survivor, i.e. multiple-survivor equilibria are *non-generic*. We are able to show that it is still possible to obtain equilibria with multiple survivors if we model the short-selling restriction, see condition (3.11), by introducing a lower and an upper bound on the values the investment function can take, thus truncating all values below (above) the lower bound (the upper bound) and setting them equal to the associated bound. If, for instance, we impose the condition $x^{\ell}_{n,t} \in [10^{-k}, 1-10^{-k}]$, $\forall n \in \mathscr{N}, \forall t \in \mathscr{T}$, and $k > 0$, trivially, if there are two traders willing to short-sell (leverage-buy) the risky asset, given their expectations, they will end up with the same (sub-)optimal investment $x^{\ell}_{n,t} = 10^{-k}$ (respectively, $x^{\ell}_{n,t} = 1 - 10^{-k}$). This assumption, for $k = 2$, was already present in the previous simulation, but it was not binding as the λ coefficient was high enough to require a larger amount of time for the equilibrium to be eventually reached than the wealth-driven selection mechanism to operate. Anufriev et al. (2006) provide a complete analysis of

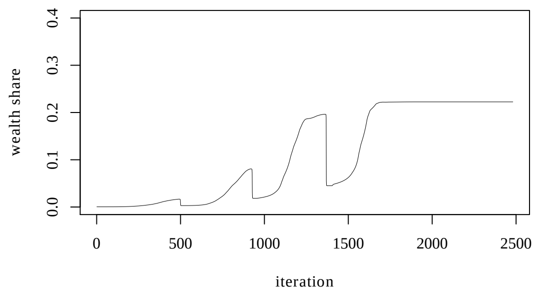
Description	Value
Initial population size	$N = 200$
Number of risky assets	$L = 1$
Static population	`true`
Riskless rate of return	$r_f = 0.02$
γ distribution	$\gamma_n \sim \mathcal{U}(1.0, 1000.0)$
λ distribution	$\lambda_n = 0.01,\ \forall n \in \mathcal{N}$
d distribution	$d_n = 1.0,\ \forall n \in \mathcal{N}$
Initial wealth endowment	$W_{n,0} = 50.0,\ \forall n \in \mathcal{N}$
Yield mean	$\bar{e} = 0.04$
Yield variance	$\sigma_e^2 = 1.0\text{e-}4$
Yield realisation distribution	$e_t \sim \mathcal{N}(\bar{e}, \sigma_e^2)$
Initial risky asset price level	$p_0 = 0.1$
x_n^ℓ admissible interval	$x_{n,t} \in [0.01, 0.99],\ \forall n \in \mathcal{N}, \forall t \in \mathcal{T}$

Table 2: Parameters and initial conditions.

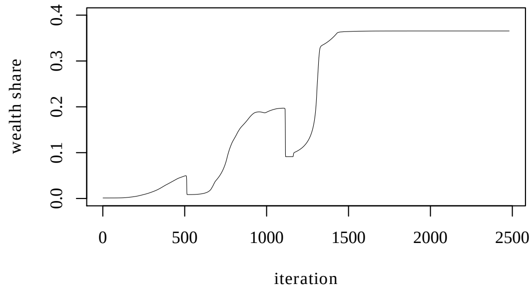
the deterministic skeleton steady-states of the system and their stability characterisation; here is sufficient to have in mind that, *cœteris paribus*, the equilibrium stability domain decreases with the value of the survivor's λ, and there exists a threshold beyond which stability is lost and neither wealth-shares, nor price, converge to a unique value. In the next simulation we propose (see Table 2 for initial conditions), we still keep parameter λ constant across the population, but we reduce its value from 0.1 to 0.01. This change makes traders revise their expectations much slower, and the magnitude of sudden price swings is much less perceived. At the time equilibrium is reached, aggregate wealth is less concentrated than before, with more than one survivor now featuring significantly positive wealth-shares. Figure 3 shows the evolution of the wealth-shares for the traders with the lowest, second-lowest, and highest γ_n, respectively corresponding to the highest, second-highest, and lowest equilibrium wealth-share φ_n^*, in two distinct runs. Plots in the left column and in the right column are almost superimposable but for an homothetic expansion, as the only difference lies in the overall riskiness of individual portfolios, driven by the risk-aversion coefficient only. Plots (e) and (f) are likewise interesting, as they show that high-risk-averse investment functions may locally outperform low-risk-averse ones: indeed, it is immediate to notice that the timing of sharp increases in the highest-γ trader's wealth-share corresponds to that of sharp decreases for low-γ traders. This result was also present in previous simulations (e.g. see again Figure 1), although it is much more evident here.

4.2 Transitional dynamics and emerging cycles

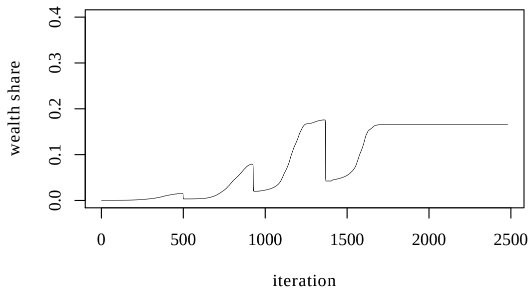
Numerical simulation, as we said, proves crucially useful in keeping track of the whole transitional dynamics of a system starting from whatever arbitrary initial conditions, and (possibly) settling into an equilibrium convergence path. In this section, we wish to analyse the evolution of the risky asset price series in order to grasp the extent to which it is explainable on the grounds of underlying fundamentals, i.e. the only exogenous component driving traders' expectations. Recall that, in our model, fundamentals are proxied by the dividend yield process, which, following **Assumption 2**, we assume roughly stationary and independent of price. We remove, for this reason, the exogenous expansionary component in eq. (3.13), letting price be uniquely determined by the evolution of the market



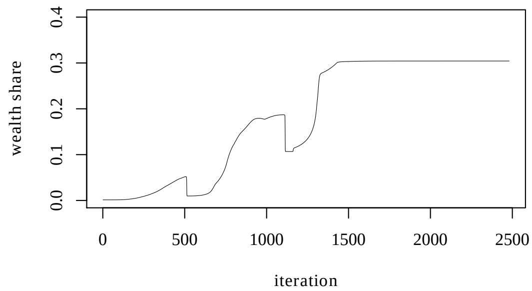
(a) Trader with lowest γ.
$\varphi^* = 0.222$



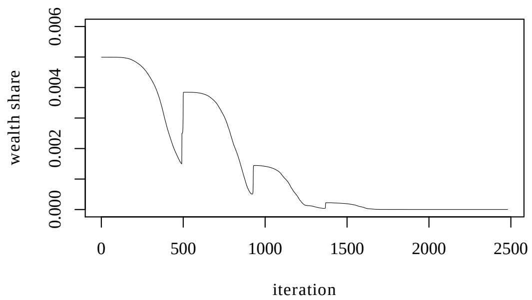
(b) Trader with lowest γ.
$\varphi^* = 0.365$



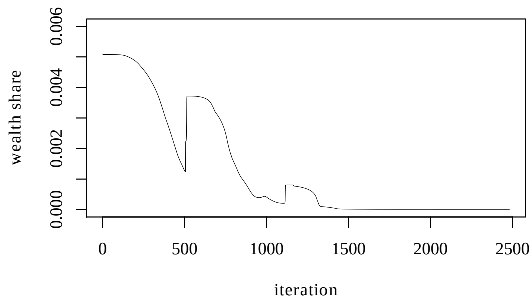
(c) Trader with second-lowest γ.
$\varphi^* = 0.166$



(d) Trader with second-lowest γ.
$\varphi^* = 0.304$



(e) Trader with highest γ.
$\varphi^* = 2.6\text{e-}6$



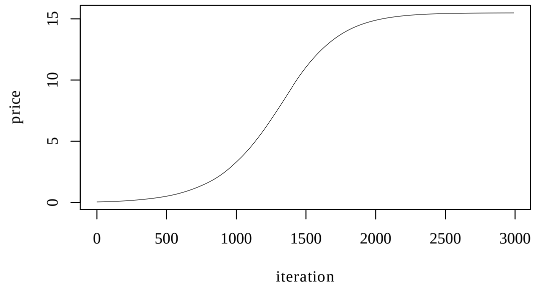
(f) Trader with highest γ.
$\varphi^* = 9.9\text{e-}6$

Figure 3: Multiple-survivor equilibrium. Evolution of wealth-shares; two typical runs (left vs. right).

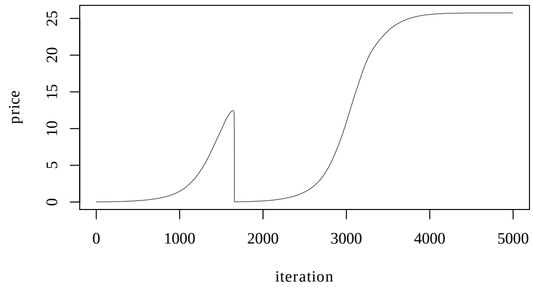
Description	Value
Initial population size	$N = 1,000$
Number of risky assets	$L = 1$
Static population	`true`
Riskless rate of return	$r_f = 0.02$
γ distribution	$\gamma_n \sim \mathcal{U}(1.0, 500.0)$
λ distribution	$\lambda_n = 0.0036, \ \forall n \in \mathcal{N}$
d distribution	$d_n = 1.0, \ \forall n \in \mathcal{N}$
Initial wealth endowment	$W_{n,0} = 50.0, \ \forall n \in \mathcal{N}$
Yield mean	$\bar{e} = 0.04$
Yield variance	$\sigma_e^2 = 1.0\texttt{e-4}$
Yield realisation distribution	$e_t \sim \mathcal{N}(\bar{e}, \sigma_e^2)$
Initial risky asset price level	$p_0 = 0.1$
x_n^ℓ admissible interval	$x_{n,t} \in [0.01, 0.99], \ \forall n \in \mathcal{N}, \forall t \in \mathcal{T}$

Table 3: Parameters and initial conditions.

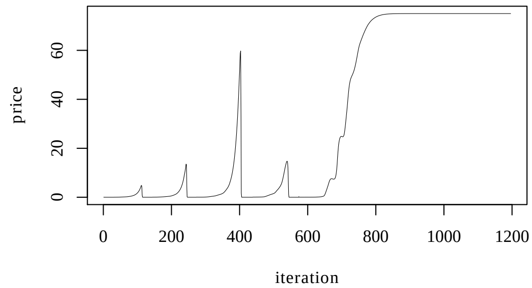
portfolio. As usual, we characterise the dynamics both in terms of the risk-aversion coefficient γ and of the memory decay factor λ. Let us start with the initial conditions listed in Table 3. There are 1,000 fully trend-chaser traders, heterogeneous in their risk-aversion coefficient, evenly endowed with the same amount of initial wealth W_0, and whose expectations updating factor is constant and set to $\lambda = 0.0036$. Resulting price dynamics is plotted in Figure 4(a). Thank to the remarkably low value of λ, the price adjusts very smoothly and in a monotonic path towards its equilibrium value $p^* = 15.48$. The particular chosen value of λ is actually a threshold: it is possible to show that for $\lambda < 0.0036$ the qualitative behaviour of price dynamics is closely similar to Figure 4(a), although the equilibrium price may vary. In particular, in the interval $(0, 0.0036]$ a monotonic relation between the memory decay factor and equilibrium price holds: the lower λ, the lower p^*, with, asymptotically, $p^* \to p_0$ as $\lambda \to 0$. By very slightly extending, *cœteris paribus*, the memory of the traders to $\lambda = 0.00365$ the adjustment path is drastically rearranged, monotonicity is broken, and equilibrium price is significantly shifted upwards. Figure 4(b) plots the new dynamics. Up to (circa) iteration 1,500, the figure proceeds identically to plot 4(a), but for the speed of adjustment which is indiscernibly greater than before, thank to the increase in λ. Such increase in the speed, however, prompts one or more high-risk-averse trader, who have not yet been completely ruled out by market selection, to sell, as the increase in the expected return variance is not completely offset by the increase in the expected value of price return. In the long run, by the way, low-risk-averse traders will increasingly dominate as showed in **Section 4.1**, and price grows up to the point selection is terminated. By further increasing the value of λ, the resulting dynamics turns more complicated, with multiple price swings which can be regarded as bubble-and-bust cycles, up to a point stability is lost, as previously hinted, and the price fluctuates indefinitely in a limit-cycle-like motion, with no convergence to an equilibrium value. This second threshold, in our experiment, is found to be approximately $\lambda \approx 0.16$, leaving a large room, within the theoretical support of the parameter, in which a clear-cut wealth-driven selection does not occur: as previously recognised, such selection mechanism is strongly subordinated to system stability, thus to the value of the trend extrapolation rate. Plots 4(c) and 4(d) show the price dynamics for $\lambda = 0.155$ and $\lambda = 0.16$, respectively, other parameters being those of Table 3. With a finite population, the exact thresholds of the parameter λ denoting the



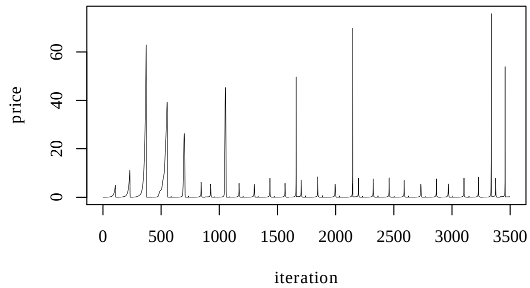
(a) Smooth convergence to equilibrium. $p^* = 15.48$



(b) Emergent fluctuation before equilibrium convergence. $p^* = 25.75$



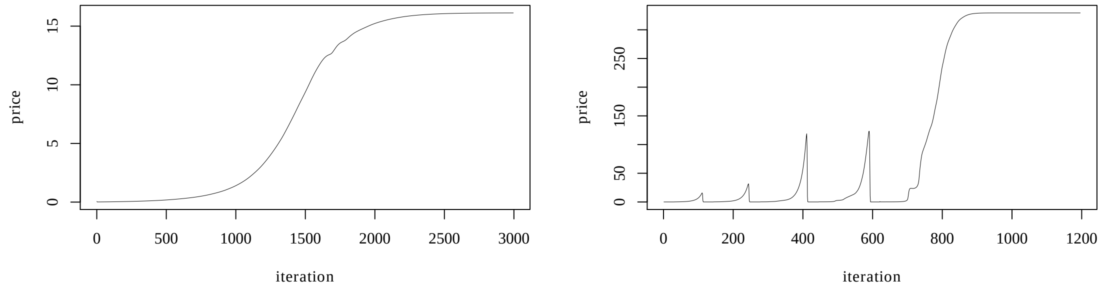
(c) Wide fluctuations before equilibrium convergence. $p^* = 75.01$



(d) No equilibrium convergence.

Figure 4: Price dynamics.

lost of monotonicity in convergence first, and overall stability after, are sensible to the way traders are initialised, that is, to the specific values of γ_n drawn from the associate distribution. Nevertheless, after performing several distinct runs, we found thresholds move away very slightly from the values outlined above. Turning now the focus on the effect of the risk-aversion coefficient on the price dynamics, we start by shrinking its support to $\gamma_n \sim \mathcal{U}(1.0, 100.0)$. Arguably, the average population risk-aversion is lower, pushing equilibrium price upwards to a value higher than before. Figure 5(a) shows such result, for $\lambda = 0.0036$ (which, in the new parameter set, is still lower than the first λ threshold), thus ensuring comparability with Figure 4(a). As expected, equilibrium price now reads $p^* = 16.13$. Another consequence of the lower population average risk-aversion is a rightward shift of the threshold where monotonicity is lost, which now holds $\lambda \approx 0.008$. Similarly, λ values above that threshold trigger price fluctuations which are now sharper than previously observed: an example is provided in Figure 5(b), which shows the price dynamics for $\lambda = 0.01$. Notably, a seemingly small variation ($\Delta\lambda = 0.0064$) of the parameter brings a 20-fold increase in equilibrium price, which now reads $p^* = 329.35$. On the other hand, following stability analysis by Anufriev et al. (2006), the second threshold is shifted leftwards, as overall riskier investing tends to destabilise the market, for a given value of λ. Values beyond this second threshold yield a plot similar to Figure



(a) Smooth convergence to equilibrium. $p^* = 16.13$

(b) Convergence with sharp fluctuations. $p^* = 329.35$

Figure 5: Price dynamics with narrower γ support.

4(d). Another emergent property we encountered in the lower-overall-γ case is that, if we introduce a second heterogeneity feature, letting λ vary across the population along with γ, a similar non-convergent dynamics is obtained, even though the support of λ lies within the stability threshold. An example is given by the case for $\lambda_n \sim \mathcal{U}(0.01, 0.1)$, when $\gamma_n \sim \mathcal{U}(1.0, 100.0)$ (recall that for $\lambda_n = 0.01$, $\forall n \in \mathcal{N}$ equilibrium price reads $p^* = 329.35$, while it is possible to show that for $\lambda_n = 0.1$, $\forall n \in \mathcal{N}$ equilibrium price exists and equals $p^* = 959.91$). This is not the case, instead if $\gamma_n \sim \mathcal{U}(1.0, 500.0)$, for which convergence still applies. This finding suggests that, if, on average, the population risk-aversion is low enough, heterogeneity in λ, i.e. in the extent to which different traders adapt to new information, may act as a destabilising force itself. On the other hand, however, it is possible to show that the case for $\gamma_n \sim \mathcal{U}(1.0, 100.0)$ is not intrinsically more *fragile* than the one for $\gamma_n \sim \mathcal{U}(1.0, 500.0)$. We investigate this point, back to the case with homogeneous λ, by introducing a random repeated shock hitting traders' investment functions: such *micro-failure* makes, every τ periods, a randomly selected trader sell his entire risky portfolio, irrespectively of his expectations about future price return. Basically, at the time the failure strikes, and for that time step only, the investment strategy of the selected trader n is forced to $x_{n,t} = 0$, implying he will lend all his wealth by means of the bond[2]. We believe such a failure can be regarded as a reasonable and likely feature of human behaviour, e.g. proxying the sudden fear of a market crash, or, speaking about financial bubbles, the belief of being the so-called *greatest fool*. Figure 6 shows the price dynamics for the same parameters of Table 3 except $\gamma_n \sim \mathcal{U}(1.0, 100.0)$, and with failure striking every $\tau = 15$ periods. During the first 1,500 iterations, the plot is largely comparable to the one in Figure 5(a), and the failure has a negligible effect, since aggregate wealth is initially evenly distributed, and market selection takes some time to operate. Gradually, with increasing wealth concentration, the influence of the failure on prevailing price becomes relevant, and the magnitude of its effect strictly depends on the actual wealth-share of the selected trader. The plot preserves an overall monotonicity, but the price fails to converge to an equilibrium, and keeps fluctuating noisily and indefinitely around a constant value, high-

[2] Notice also that, as long as just one or a few traders are concerned, sufficient condition (3.11) may not be satisfied, but results of **Proposition 1** hold anyway.

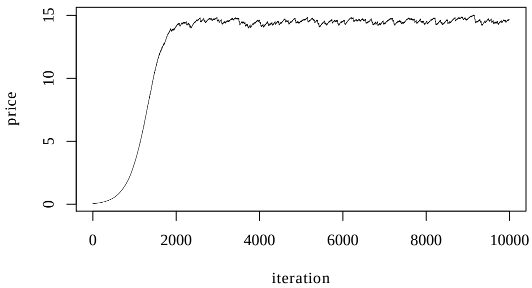


Figure 6: Price dynamics with micro-failure striking every $\tau = 15$ periods; risk-aversion coefficient $\gamma_n \sim \mathcal{U}(1.0, 100.0)$

lighting the fact that wealth-driven selection is eventually overwhelmed by the shock and fails to operate from a certain period afterwards. Applying the very same behavioural shock to the case for $\gamma_n \sim \mathcal{U}(1.0, 500.0)$ and other parameters equal, yields a significantly different transient: see Figure 7. Up to circa iteration 1,500 the reasoning holds as usual;

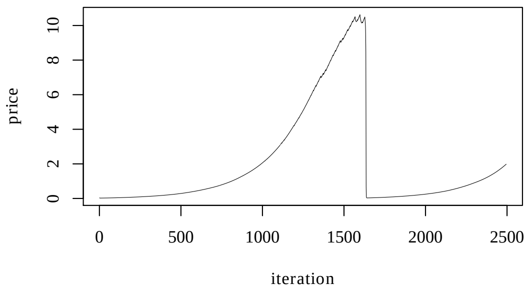


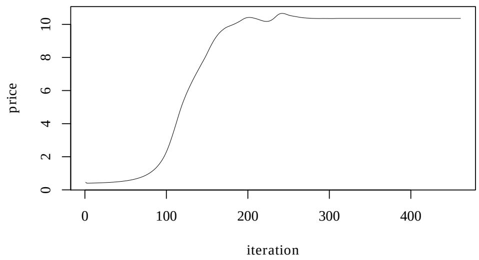
Figure 7: Price dynamics with micro-failure striking every $\tau = 15$ periods; risk-aversion coefficient $\gamma_n \sim \mathcal{U}(1.0, 500.0)$

then, as soon as an enough powerful trader (in terms of wealth-share) is hit by the failure, the sudden (although limited) plunge in the price prompts the most risk-averse traders (who are now more averse than before) to sell the security during subsequent iterations, whose dropping price will in turns push other traders towards *safer* positions, eventually triggering a 'selling spree', in which the price eventually spirals downwards up to the point expected variance is low enough to convince investors of taking riskier positions. The cycle then repeats itself as long as the failure keeps striking. It is out of the scope of this paper to provide a precise estimate of the various thresholds involved. It is of crucial interest, instead, showing that the emergent properties observed, i.e. the bubble-and-bust cycles shown in Figures 4(b), 4(c) and 5(b), preserve robustness with respect to the introduction of fundamentalist and trend-contrarian traders in the economy. On the one hand, the presence of fundamentalists is expected to stabilise the price of an asset, as they act against chartists whenever current price deviates from its fundamental value.

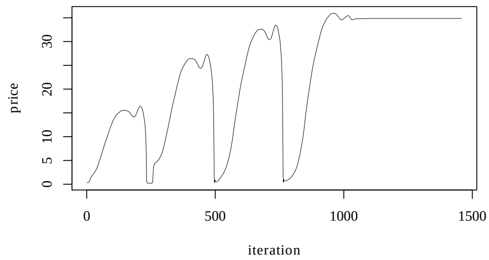
Description	Value
Initial population size	$N = 1,000$
Number of risky assets	$L = 1$
Static population	`true`
Riskless rate of return	$r_f = 0.02$
γ distribution	$\gamma_n \sim \mathcal{U}(1.0, 500.0)$
λ distribution	$\lambda_n = 0.05,\ \forall n \in \mathscr{N}$
d distribution	$d_n \sim \mathcal{N}(0, 1)$
Initial wealth endowment	$W_{n,0} = 50.0,\ \forall n \in \mathscr{N}$
Yield mean	$\bar{e} = 0.04$
Yield variance	$\sigma_e^2 = 1.0\text{e-}4$
Yield realisation distribution	$e_t \sim \mathcal{N}(\bar{e}, \sigma_e^2)$
Initial risky asset price level	$p_0 = 0.1$
x_n^ℓ admissible interval	$x_{n,t} \in [0.01, 0.99],\ \forall n \in \mathscr{N}, \forall t \in \mathscr{T}$

Table 4: Parameters and initial conditions.

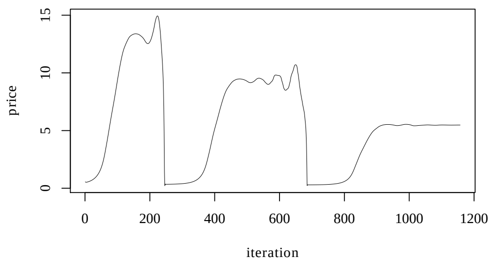
On the other hand, trend-contrarians shall offset the attempt made by trend-chasers to exacerbate the price trend, by acting in a symmetrical manner with respect to these latter. Following Chiarella and He (2001), we exploit the d parameter in eq. (3.22) to differentiate the agents with respect to their trading approach. Parameters and initial conditions are listed in Table 4. In particular, we assume d_n to be normally distributed, meaning that most of traders are quasi-fundamentalists, as probability density is concentrated around the (zero) mean value, and that chartists are well balanced between trend-chasers and contrarians, as the density is symmetric around the mean. If the memory of the traders is short enough, the result is pretty similar to the trend-chasers-only case: compare Figure 8(a), simulated for $\lambda = 0.05$, with Figures 4(a) and 5(a), and notice also that the presence of fundamentalists let monotonicity be preserved for a larger-than-before extrapolation rate. By increasing the value of λ, as previously obtained, price dynamics suffers large swings before settling towards an equilibrium, even though fluctuations have a different shape and look less sharp: see Figure 8(b), simulated for $\lambda = 0.1$. Setting different support for the risk-aversion coefficient yields results similar to the constant $d = 1.0$ case, with a negative relation between the population average risk-aversion and equilibrium price: see e.g. Figure 8(c), obtained for $\lambda = 0.1$ and $\gamma_n \sim \mathcal{U}(1.0, 1000.0)$, featuring a lower equilibrium price than in Figure 8(b). Notice also that, with heterogeneity applying to both γ and d, the exact value of equilibrium price may be influenced, especially when dealing with small populations, by the random process initialising traders' behavioural parameters: for instance, it can happen that a trader featuring a quite low risk-aversion coefficient also has a d very close to zero, or that a strong trend-chaser might be, at the same time, very risk-averse. By the way, as long as we are interested in the evolution of transitional dynamics, rather than the specific final price, simulations with both chartists and fundamentalists do nothing else than add robustness to the previous findings concerning the decoupling of the price dynamics from the dividend yield process we obtained in the all-trend-chasers case.



(a) Smooth convergence towards equilibrium. $\lambda = 0.05$, $p^* = 10.36$



(b) Convergence with fluctuations. $\lambda = 0.1$, $p^* = 34.86$



(c) Larger γ support. Convergence with fluctuations. $\lambda = 0.1$, $p^* = 5.47$

Figure 8: Price dynamics; fundamentalists vs. chartists.

5 Concluding remarks and further research

We study an asset pricing model where adapting heterogeneous traders take portfolio decisions as smooth function of their Exponentially Weighted Moving Average estimates of future asset return and variance. Expectations are based uniquely on past market history and on the risky asset dividend yield, which we intend as a proxy for fundamentals. The assumptions we impose include the yield process being governed by a stationary distribution, and short-selling restriction on investment. Time is discrete, and at the end of each period market clearing price is announced by a fictional walrasian auctioneer. Traders are mean-variance myopic optimisers and have CRRA attitude, implying that their invested shares of wealth do not depend on their wealth level, or equivalently, that individual demand functions are proportional to individual wealth. We set up a numerical simulation of the system, featuring one risky and one riskless asset, and an arbitrarily large population of traders who differ in their relative risk-aversion coefficient, their memory decay factor in the EWMA estimators, and their trading strategy, i.e. they can act as fundamentalists or chartists (either trend-chasers or contrarians). While we mainly focus on theoretical aspects and seek no particular fit into actually observed financial time-series, our results are intended as a contribution to the growing literature on artificial agent-based stock markets. By analysing the evolution of individual wealth-shares, we are able to replicate some of the findings already present in previous literature concerning the asymptotic dominance of different investment functions. In particular we show that, in a all-chartist framework, either single- and multiple-survivor equilibria are possible, largely depending on the parameter regulating traders' memory: in particular, when traders present long backward-looking horizon in forming expectations, a strong wealth-driven selection mechanism applies, and only the most-aggressive (e.g. the least-risk-averse) trader asymptotically survives and therefore dominates the economy. Reducing such horizon makes the selection process operate only partially, and more than one trader eventually survives. When, instead, traders take into account only a few last observations, the system loses stability and there is no convergence to an equilibrium outcome. Coherently, shifting the focus on the price dynamics, we show that, if traders' memory is homogeneous across the population and is long enough, the price of the risky security smoothly adjusts and monotonously converge to an equilibrium value, irrespectively of the heterogeneity in risk-aversion (captured by the width of the associated coefficient distribution support). Reducing the memory beyond a certain threshold, but still assuming values which we repute rather reasonable for a technical analyst attempting to extrapolate the historical trend, brings a dramatic change in the overall dynamics, triggering the emergence of one or more bubble-and-bust-like cycles before the system settles into an equilibrium convergence path. A similar rich dynamics is also obtained in the longer-memory framework in two different experiments: i) by introducing memory heterogeneity among the traders, if average population risk-aversion is sufficiently low; ii) by introducing a micro-failure, repeatedly (even loosely) hitting traders' investment decisions, if risk-aversion coefficient distribution support is wide enough; in this case, however, the equilibrium is not reached as long as the shock keeps striking. These experiments make us conclude that both heterogeneity in risk-aversion, and low overall risk-aversion may singularly produce a destabilising effect. As before, with exceedingly *naïve* expectations, stability is lost and the price fluctuates indefinitely in a limit-cycle-like motion. Finally, we extend the experiment displaying bubble-and-bust transitional dynamics, by adopting a different population, largely composed by quasi-fundamentalists and well balanced crowds of trend-chasers and contrarians. In spite of the existence of fundamentalists and contrarians, who are generally expected to counteract the aggressiveness of trend-chasers, the emergence of booms and crashes is validated, though fluctuations may exhibit different shape or amplitude, adding further robustness to our findings.

To sum up, our model is able to show that, for all-but-unreasonable values of traders' memory, short-selling restriction and risk-aversion heterogeneity are sufficient conditions to trigger a sharp decoupling of the price dynamics from the fundamental process, driven by a market selection mechanism that rewards the least-risk-averse traders and pushes the system towards higher than fundamental equilibria. During transition, a very rich dynamics is obtained without further assumptions, with emergent bubble-and-bust cycles arising uniquely from the market interaction of the traders involved. On the other hand, our model is not able to replicate the whole ensemble of stylised facts observable in real financial markets.

Our framework can be further extended in a number of directions. A straightforward improvement is to distinctly account for multiple (instead of just one, or an index of) risky assets. This allows to study the cross-correlation structure of different assets returns and to investigate the conditions prompting the emergence of excess covariance. Perhaps more interestingly, the model can be enriched by implementing more realistic methods in decision-making, such as those prescribed by *prospect theory* (see Kahneman and Tversky, 1979), rule-based techniques, articulated learning processes (e.g. through classifier systems or genetic algorithms), and herd behaviour. Our guess is that a sharper departure from rationality assumptions and, relatedly, a more structured modelling of agents' behaviour are required in order to obtain dynamics that are closer to reality.

References

Anufriev, M. and G. Bottazzi (2010). "Market Equilibria under Procedural Rationality". In: *Journal of Mathematical Economics* 46.6, pp. 1140–1172.

Anufriev, M., G. Bottazzi, M. Marsili, and P. Pin (2012). "Excess covariance and dynamic instability in a multi-asset model". In: *Journal of Economic Dynamics and Control* 36.8, pp. 1142–1161.

Anufriev, M., G. Bottazzi, and F. Pancotto (2006). "Equilibria, stability and asymptotic dominance in a speculative market with heterogeneous traders". In: *Journal of Economic Dynamics and Control* 30.09–10, pp. 1787–1835.

Anufriev, M. and P. Dindo (2010). "Wealth-driven Selection in a Financial Market with Heterogeneous Agents". In: *Journal of Economic Behavior and Organization* 73, pp. 327–358.

Arthur, W. B., J. H. Holland, B. LeBaron, and R. G Palmer (1997). "Asset Pricing Under Endogenous Expectation in an Artificial Stock Market". In: *The Economy as an Evolving Complex System II*. Ed. by W. B. Arthur, S. Durlauf, and D. Lane. Addison-Wesley.

Black, F. and M. Scholes (1973). "The Pricing of Options and Corporate Liabilities". In: *Journal of Political Economy* 81.3, pp. 637–654.

Brock, W. and C. Hommes (1998). "Heterogeneous Beliefs and Routes to Chaos in a Simple Asset Pricing Model". In: *Journal of Economic Dynamics and Control* 22.8–9, pp. 1235–1274.

Chiarella, C. and X. He (2001). "Asset price and wealth dynamics under heterogeneous expectations". In: *Quantitative Finance* 1, pp. 509–526.

Colander, D., H. Follmer, A. Haas, M. Goldberg, K. Juselius, A. Kirman, T. Lux, and B. Sloth (2009). "The Financial Crisis and the Systemic Failure of the Economics Profession". In: *Critical Review* 21.2, pp. 249–267.

Cont, R. (2001). "Empirical properties of asset returns: stylized facts and statistical issues". In: *Quantitative Finance* 1, pp. 223–236.

Cutler, D., J. Poterba, and L. Summers (1989). "What moves stock prices?" In: *Journal of Portfolio Management* 15, pp. 4–12.

Ding, Z., C. W. J. Granger, and R. F. Engle (1993). "A long memory property of stock market returns and a new model". In: *Journal of Empirical Finance* 1, pp. 83–106.

Fama, E. F. (1970). "Efficient Capital Markets: A Review of Theory and Empirical Work". In: *The Journal of Finance* 25.2, pp. 383–417.

Fama, E. F. (1991). "Efficient Capital Markets: II". In: *The Journal of Finance* 46.5, pp. 1575–1617.

Frankel, J. A. and K. A. Froot (1986). "Understanding the US dollar in the eighties: the expectations of chartists and fundamentalists". In: *Economic Record* 1.2, pp. 24–38.

Froot, K. A. and E. M. Dabora (1999). "How are stock prices affected by the location of trade?" In: *Journal of Financial Economics* 53.2, pp. 189–216.

Grandmont, J. M. (1985). "On Endogenous Competitive Business Cycles". In: *Econometrica* 53.5, pp. 995–1045.

Guillaume, D. M., M. M. Dacorogna, R. R. Davé, U. A. Müller, R. B. Olsen, and O. V. Pictet (1997). "From the bird's eye to the microscope: A survey of new stylized facts of the intra-daily foreign exchange markets". In: *Finance and Stochastics* 1.2, pp. 95–129.

Hommes, C. (2006). "Heterogeneous Agent Models in Economics and Finance". In: *Handbook of Computational Economics, Volume 2*. Ed. by L. Tesfatsion and K. Judd. North Hol-

land.
Kahneman, D. and A. Tversky (1979). “Prospect Theory: An Analysis of Decision under Risk”. In: *Econometrica* 47.2, pp. 263–291.
Kirman, A. P. (1992). “Whom or What does the Representative Agent represent?” In: *Journal of Economic Perspectives* 6.2, pp. 117–136.
LeBaron, B. (2006). “Agent-Based Computational Finance”. In: *Handbook of Computational Economics, Volume 2*. Ed. by L. Tesfatsion and K. Judd. North Holland.
LeBaron, B., W. B. Arthur, and R. G Palmer (1999). “Time series properties of an artificial stock market”. In: *Journal of Economic Dynamics and Control* 23.9–10, pp. 1487–1516.
LeRoy, S. F. and R. D. Porter (1981). “The present value relation: Tests based on implied variance bounds”. In: *Econometrica* 49, pp. 555–574.
Levy, H. (1994). “Absolute and Relative Risk Aversion: An Experimental Study”. In: *Journal of Risk and Uncertainty* 8, pp. 289–307.
Levy, M. and H. Levy (1996). “The Danger of Assuming Homogeneous Expectations”. In: *Financial Analysts Journal* 52.3, pp. 65–70.
Levy, M., H. Levy, and S. Solomon (1994). “A Microscopic Model of the Stock Market: Cycles, Booms, and Crashes”. In: *Economics Letters* 45.1, pp. 103–111.
Levy, M., H. Levy, and S. Solomon (1995). “Simulation of the Stock Market: The Effects of Microscopic Diversity”. In: *Journal de Physique I* 5.8, pp. 1087–1107.
Levy, M., H. Levy, and S. Solomon (2000). *Microscopic Simulation of Financial Markets*. Academic Press, London.
Levy, M., N. Persky, and S. Solomon (1996). “The Complex Dynamics of a Simple Stock Market Model”. In: *International Journal of High Speed Computing* 8.93, pp. 93–119.
Mandelbrot, B. (1963). “The Variation of Certain Speculative Prices”. In: *Journal of Business* 36.4, pp. 394–419.
Merton, R. C. (1973). “Theory of Rational Option Pricing”. In: *The Bell Journal of Economics and Management Science* 4.1, pp. 141–183.
Palmer, R. G., W. B. Arthur, J. H. Holland, B. LeBaron, and P. Tayler (1994). “Artificial Economic Life: a Simple Model of a Stockmarket”. In: *Physica D* 75, pp. 264–274.
Pindyck, R. S. and J. Rotemberg (1993). “The Comovement of Stock Prices”. In: *Quarterly Journal of Economics* 108.4, pp. 1073–1104.
Ribeiro, R. and P. Veronesi (2002). *The Excess Comovement of International Stock Markets in Bad Times: A Rational Expectations Equilibrium Model*. Mimeo.
Roubini, N. (2006). “Why Central Banks Should Burst Bubbles”. In: *International Finance* 9.1, pp. 87–107.
Schiff, P. D. and J. Downes (2007). *Crash Proof: How to Profit from the Economic Collapse*. Wiley.
Shiller, R. J. (1981). “Do stock prices move too much to be justified by subsequent changes in dividends?” In: *American Economic Review* 71, pp. 421–436.
Shiller, R. J. (1989). *Market Volatility*. MIT Press, Cambridge.
Stiglitz, J. E. (1990). “Symposium on Bubbles”. In: *Journal of Economic Perspectives* 4.2, pp. 13–18.
Tesfatsion, L. and K. L. Judd (2006). *Handbook of Computational Economics, Volume 2*. North Holland.
Zschischang, E. and T. Lux (2001). “Some new results on the Levy, Levy and Solomon microscopic stock market model”. In: *Physica A: Statistical Mechanics and its Applications* 291.1–4, pp. 563–573.

Appendix A

Proof of **Proposition 1.** Once the individual wealth evolution is defined in equation (3.5), aggregate wealth holds:

$$W_t = W_{t-1} \cdot \left[x^0_{t-1} \cdot (1 + r_f) + \sum_{\ell=1}^{L} x^\ell_{n,t-1} \cdot \left(\frac{p^\ell_t}{p^\ell_{t-1}} + e^\ell_t \right) \right] \tag{A.1}$$

Substituting (A.1) into pricing equation (3.10) yields

$$\frac{p^\ell_t}{p^\ell_{t-1}} = \frac{x^\ell_t \cdot W_t}{x^\ell_{t-1} \cdot W_{t-1}} = \frac{x^\ell_t}{x^\ell_{t-1}} \cdot \left[x^0_{t-1} \cdot (1 + r_f) + \sum_{\ell=1}^{L} x^\ell_{n,t-1} \cdot \left(\frac{p^\ell_t}{p^\ell_{t-1}} + e^\ell_t \right) \right] \tag{A.2}$$

where asset prices still appear in both LHS and RHS. If we multiply both sides by x^ℓ_{t-1} and sum over ℓ, we get

$$\sum_{\ell=1}^{L} x^\ell_{t-1} \cdot \frac{p^\ell_t}{p^\ell_{t-1}} = \sum_{\ell=1}^{L} x^\ell_t \cdot \left[x^0_{t-1} \cdot (1 + r_f) + \sum_{\ell=1}^{L} x^\ell_{n,t-1} \cdot \left(\frac{p^\ell_t}{p^\ell_{t-1}} + e^\ell_t \right) \right] \tag{A.3}$$

We are now able to compute the LHS in terms of known quantities:

$$\sum_{\ell=1}^{L} x^\ell_{t-1} \cdot \frac{p^\ell_t}{p^\ell_{t-1}} = \frac{1 - x^0_t}{x^0_t} \cdot \left[x^0_{t-1} \cdot (1 + r_f) + \sum_{\ell=1}^{L} x^\ell_{n,t-1} \cdot e^\ell_t \right] \tag{A.4}$$

Substituting (A.4) into (A.1) yields the implied evolution of aggregate wealth

$$W_t = W_{t-1} \cdot \frac{x^0_{t-1} \cdot (1 + r_f) + \sum_{\ell=1}^{L} x^\ell_{n,t-1} \cdot e^\ell_t}{x^0_t} \tag{A.5}$$

the knowledge of which, along with (3.10) yields the equilibrium prevailing price:

$$p^\ell_t = p^\ell_{t-1} \cdot \frac{x^\ell_t}{x^\ell_{t-1}} \cdot \frac{x^0_{t-1} \cdot (1 + r_f) + \sum_{\ell=1}^{L} x^\ell_{n,t-1} \cdot e^\ell_t}{x^0_t} \tag{A.6}$$

The LHS is clearly strictly positive if

$$x^\ell_t > 0 \qquad \forall \ell \in \mathscr{L}, \forall t \in \mathscr{T} \tag{A.7}$$

or, equivalently, if

$$x^\ell_{n,t} \in (0, 1) \qquad \forall n \in \mathscr{N}, \forall \ell \in \mathscr{L}, \forall t \in \mathscr{T} \tag{A.8}$$

□