Economic Growth
Chapter 6 : Endogenous saving behavior, and optimal growth

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Consumption-saving trade-off

- Saving does not result from a Keynesian behavior
- But from intertemporal *arbitrage* between present and future consumption by infinitely living households.
- A very simple economy, composed of homogenous households, producing a unique good using a constant-returns to scale production technology.
- The economy's behavior is determined by a central planner, trying to maximize the social welfare (the utility of the representative household.

- The household's size (N_t) increases at rate n at each instant).
- Each member of the household offers one unit of labor (labor supply $= N_t$).
- The good is produced using labor and capital (K_t):

$$Y_t = F(K_t, N_t) = C_t + \dot{K}_t \tag{1}$$

  the share of the product that is not used as consumption (C_t) is invested.
- $K_0 > 0$. No depreciation of capital.
- $F(\cdot)$ is neoclassical.
- In per capita terms

$$f(k) = c + \dot{k} + nk \Leftrightarrow \dot{k} = f(k) - c - nk, k_0 > 0, f' >, f'' < 0 \tag{2}$$

Consumption and welfare

- At each moment s, the utility of the household is given, in current value, by:

$$u(c_s) \geq 0, \; u' \geq 0, \; u'' \leq 0, \tag{3}$$

(the utility function of the household).

- And the current value, in period s, of a future consumption, at a date $t > s$, is given by

$$u(c_t) \cdot e^{-\theta(t-s)}$$

where $1 > \theta > 0$ is the subjective discount factor of the household, resulting from its time preference.

- The lifetime utility of the household, evaluated at the start of its life is hence given by:

$$U_0 = \int_0^{\infty} u(c_t) \cdot e^{-\theta t} \cdot dt \tag{4}$$

Optimal consumption

- The consumption path that would maximize the total welfare of the households
- taking into account the characteristics of the economy and
- setting a favorable consumption-saving trade-off at each moment of time, warranting an optimal investment path.

- The planner must hence solve the following problem

$$\max_{c_t} U_0 = \int_0^\infty u(c_t) \cdot e^{-\theta t} \cdot dt \tag{5}$$

$$\text{with} \quad \begin{cases} \dot{k}_t = f(k_t) - c_t - nk_t \\ k(0) = k_0 \\ \forall t, \ k_t \geq 0, c_t \geq 0 \end{cases} \tag{6}$$

- in order to determine the optimal consumption path $c^*(t) = c_t^*$.
- We have dynamic control problem.
- The control variable is c_t, and the state variable is k_t.
- The *motion* in time of the state variable is constrained by the first condition of (6) (*state equation*).

The Hamiltonian

- We can solve this problem by using the optimal control principle of Pontryagin.
- We should write a Hamiltonian by penalizing the objective by the violation of the constraint on the evolution of the state variable, using a shadow price (or *costate* variable), μ_t, for this penalization:

$$H_t = u(c_t)\, e^{-\theta t} + \mu_t \cdot \left(\underbrace{f(k_t) - c_t - nk_t}_{\dot{k}} \right) \tag{7}$$

- This shadow price corresponds to the **marginal value**, at moment 0, of of one supplementary unit of capital at moment t, for the household's utility (if we could marginally relax this constraint at moment t).

- It is easier to work with the current value, at moment t, of this marginal value:

$$\lambda_t \equiv \mu_t \cdot e^{\theta t} \Leftrightarrow \mu_t = \lambda_t \cdot e^{-\theta t} \tag{8}$$

$$\Rightarrow \dot{\lambda} = \dot{\mu} \cdot e^{\theta t} + \theta \mu \cdot e^{\theta t} = \dot{\mu} \cdot e^{\theta t} + \theta \lambda \tag{9}$$

- and, by replacing μ by (8), the Hamiltonian can be written as:

$$H_t = e^{-\theta t} \cdot \left[u\left(c_t\right) + \lambda_t \cdot \left(f\left(k_t\right) - c_t - nk_t\right)\right] \tag{10}$$

- If we neglect for now positivity conditions, and given the properties of u and f, the sufficient conditions of optimality are given by:

$$\frac{\partial H}{\partial c} = 0 \tag{11}$$

$$\frac{d\mu}{dt} \equiv \dot{\mu} = -\frac{\partial H}{\partial k} \tag{12}$$

$$\lim_{t \to \infty} k_t \mu_t = 0 \tag{13}$$

- (11) is the optimality condition for c, (12) is the standard motion equation of the costate variable, and (13) is the transversality condition (more on this later).

If we use the current value version of H, (10),using λ, the previous conditions become:

$$\frac{\partial H}{\partial c} = u'(c_t) - \lambda_t = 0 \Rightarrow \lambda_t = u'(c_t) \tag{14}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial k} \cdot e^{\theta t} + \theta\lambda = \lambda \cdot (n - f'(k) + \theta) \tag{15}$$

$$\lim_{t\to\infty} k_t \cdot \left(\lambda_t \cdot e^{-\theta t}\right) = \lim_{t\to\infty} k_t \cdot \left(u'(c_t) \cdot e^{-\theta t}\right) = 0 \tag{16}$$

Equations (14) and (15) can be combined for eliminating the costate variable λ:

$$\dot{\lambda} = \frac{du'(c_t)}{dt} = u'(c_t) \cdot (n - f'(k) + \theta)$$

$$\Rightarrow \frac{du'(c_t)/dt}{u'(c_t)} = n - f'(k) + \theta \tag{17}$$

$$\Leftrightarrow \frac{u''(c_t) \cdot (dc_t/dt)}{u'(c_t)} = n - f'(k) + \theta$$

$$\Leftrightarrow \left[\frac{c_t \cdot u''(c_t)}{u'(c_t)}\right] \cdot \frac{\dot{c}_t}{c_t} = n - f'(k) + \theta \tag{18}$$

Finally

- The determining conditions hence are (18) et (16) :

$$\left[\frac{c_t \cdot u''}{u'}\right] \frac{\dot{c}_t}{c_t} = n - f'(k) + \theta$$
$$\lim_{t \to \infty} k \cdot \left(\lambda_t \cdot e^{-\theta t}\right) = \lim_{t \to \infty} k \cdot \left(u'(c_t) \cdot e^{-\theta t}\right).$$

- The first one must me respected along the complete trajectory, and it is called in the literature, the *Keynes-Ramsey rule*.

In discrete time

- The intuition behind this condition could be better understood, if we consider the time allocation of consumption problem in discrete time (Keynes), between periods t and $t+1$
- If we reduce the consumption c_t of dc, the household's utility in t will be diminished of $u'(c_t)\,dc$.
- But a higher investment will be obtained at t, and a higher consumption at $t+1$: $df = dc\,(1+f'(k))$.
- Since the population will increase in the mean time, the per capita consumption in $t+1$ can only be increased of

$$\frac{dc \cdot (1+f'(k))}{1+n}.$$

- Which would imply a utility increase in $t+1$, evaluated in t of

$$\frac{1}{1+\theta} \cdot u'(c_{t+1}) \cdot \frac{dc \cdot (1+f'(k))}{1+n}.$$

- Since we are initially on the optimal BGP, par assumption, that intertemporal substitution should not increase the total welfare of the household:

$$\begin{aligned} u'(c_t)\,dc &= \frac{1}{1+\theta}\cdot u'(c_{t+1})\cdot\frac{dc\cdot(1+f'(k))}{1+n} \\ &\Rightarrow u'(c_t) = \frac{1}{1+\theta}\cdot u'(c_{t+1})\cdot\frac{1+f'(k)}{1+n} \\ &\Leftrightarrow \frac{\frac{1}{1+\theta}\cdot u'(c_{t+1})}{u'(c_t)} = \frac{1+n}{1+f'(k)} \qquad (19) \\ &\Leftrightarrow MSR_{t+1,t} = MTSR_{t+1,t} \end{aligned}$$

which would be equivalent to condition (18) for infinitesimal dt.

- $\rightarrow$ The future consumption should be increased/constant/decreased if the marginal product of the capital (net of population growth) is higher/equal/lower that the time preference rate of the household: in this case, it is preferable to reduce the current consumption for increasing the future one (intertemporal elasticity).

Desired behavior of the system at the horizon

- It guides the selection of the optimal path in accordance (transversality condition)
- In our case the horizon is infinite, and the condition is given by (16):

$$\lim_{t\to\infty} k_t \cdot \left(\lambda_t \cdot e^{-\theta t}\right) = \lim_{t\to\infty} k_t \cdot \left(u'\left(c_t\right) \cdot e^{-\theta t}\right) = 0.$$

- We can better understand its signification if we consider a finite horizon T.
- At this horizon, if $u'\left(c_T\right) \cdot e^{-\theta T}$ was positive, it would not be optimal to finish with a positive capital stock (we would use it to increase consumption).

- Consequently, at the end of the optimal path, we should have

$$k_T > 0 \text{ and } u'(c_T^*) \cdot e^{-\theta T} = 0$$
$$\text{or}$$
$$k_T = 0 \text{ and } u'(c_T^*) \cdot e^{-\theta T} > 0$$
$$\text{or}$$
$$k_T = 0 \text{ and } u'(c_T^*) \cdot e^{-\theta T} = 0$$

- Conditions that we can combine as a unique one:

$$k_T \cdot \left(u'(c_T) \cdot e^{-\theta T}\right) = 0$$

- which would become, with a very far horizon

$$\lim_{T \to \infty} k_T \cdot \left(u'(c_T) \cdot e^{-\theta T}\right) = 0.$$

Optimal BGP

- Determined by the conditions:

$$\dot{k} = f(k) - c - nk, \quad (\leftarrow 2) \tag{20}$$

$$\dot{c} = \frac{u'(c_t)}{u''(c_t)} \cdot \left(n - f'(k) + \theta\right), \quad (\leftarrow 17) \tag{21}$$

$$\lim_{t \to \infty} k_t \cdot \left(u'(c_t) \cdot e^{-\theta t}\right) = 0. \tag{22}$$

- And on the BGP, we should have constant k and $c \Rightarrow \dot{c} = 0, \dot{k} = 0$.

Dynamics of the economy

- In order to characterize these dynamics, we will build the phase diagram of this economy, in the frame (k, c).
- Dynamics of the economy are given by the equations(20) et (21):

$$\dot{k} = f(k) - c - nk, \qquad (\leftarrow 2) \tag{23}$$

$$\dot{c} = \frac{u'(c_t)}{u''(c_t)} \cdot \left(n - f'(k) + \theta\right), \ (\leftarrow 17) \tag{24}$$

- On the BGP:

$$\dot{c} = 0 \Rightarrow f'(k) = n + \theta \Rightarrow k = k^* \ \text{(constant)} \tag{25}$$

  a vertical line at $k = k^*$.
- And

$$\dot{k} = 0 \Rightarrow c = f(k) - nk \quad \text{(fonction of } k\text{)}$$

  which attain a maximum for the Golden Rule capital/worker of the Solow model: $k = k_{or}$ $(f'(k_{or}) = n)$. It starts at $(0, 0)$, and becomes 0 for $k = A$, $(f(A) = n.A)$.

Dynamics around the BGP

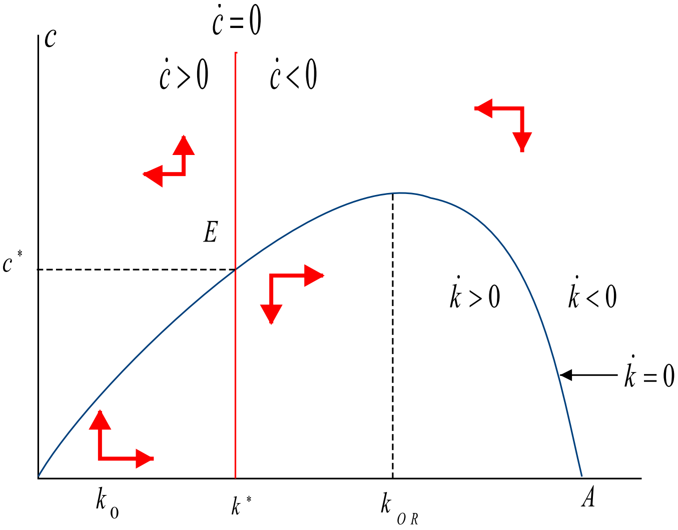
- Around $\dot{c} = 0$:

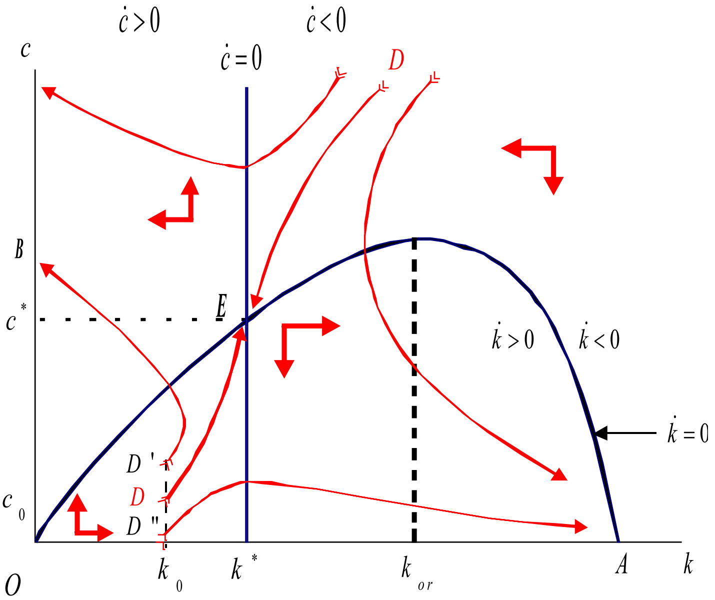
$$\dot{c} = 0 \Leftrightarrow f'(k^*) - n - \theta = 0$$
$$k < k^* \Rightarrow f'(k) > f'(k^*) = n + \theta \Rightarrow \dot{c}(k) > 0$$

- Around $\dot{k} = 0$:

$$\dot{k} = 0 \Leftrightarrow f(k^*) - c^* - nk^* = 0$$
$$c > c^* \Rightarrow \dot{k}(k^*, c) = f(k^*) - c - nk^* < 0$$

Phase diagram




$\dot{c} > 0$
$\dot{c} < 0$
c
$\dot{c} = 0$
D
B
c^*
E
$\dot{k} > 0$
$\dot{k} < 0$
$\dot{k} = 0$
D'
D
c_0
D''
k_0
k^*
k_{or}
A
k
O

Equilibria and stability

- Three equilibria: The origin O, the point E, and the point A.
- On all trajectories, except DD, either the condition Keynes-Ramsey, or the transversality condition is violated.
- DD is the only trajectory that respects all conditions, and it converges on the equilibrium E: it is the stable branch of evolution.
- If we start above D (eg. point D'), the capital stock becomes nil and we must jump to the origin (excluded by the equation of consumption dynamics).
- If we start below D (eg. point D''), we converge to point A, and end up with a positive stock of capital (excluded by the transversality condition).
- For any initial k_0, we can find a level of consumption that puts the economy on this branch.
- This trajectory is optimal trajectory.

So we do not respect the Golden rule, do we?

- From equation (21), with $\dot{c} = 0$, we observe that the optimal path would imply:

$$f'(k^*) = n + \theta \Rightarrow k^* = f'^{-1}(n + \theta) \tag{26}$$

- Or, Solow's Golden Rule would correspond to $f'(k_{or}) = n$.
- Now this rule gives us the maximal level for the capital stock on the BGP:

$$\begin{gathered} f'(k^*) = n + \theta > n = f'(k_{or}) \\ f'' < 0 \Rightarrow k^* < k_{or}, \\ f(k^*) < f(k_{or}),\ c^* < c_{or}. \end{gathered}$$

- but not the optimal level, which is lower.
- Even if the household could consume more, because of the time preference (θ), it would not be optimal to reduce more the current consumption, in order to attain, in the future, the higher level corresponding to the Golden rule.

- This result is quite strong:
- In fine, the productivity of the capital, and the real interest rate are determined by the time preference and population growth.
- The technology then determines the capital stock and the consumption level compatible with this real interest rate.
- It is also possible to show that a decentralized version of this economy, with perfectly competitive capital and labor markets, and debts for transferring purchasing power in time, would result in the same BGP, and welfare properties, under perfect foresight.
- This centralized optimal BGP could be decentralized by a market system.